5.4.4. Let $g(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be any polynomial. Since W is a T-invariant subspace of V, we know that $T^n(W) \subseteq T^{n-1}(W) \subseteq \cdots \subseteq T(W) \subseteq W$. Let $w \in W$, then

$$g(T)w = a_n T^n(w) + a_{n-1}T^{n-1}(w) + \cdots + a_1 T(w) + a_0 w \in W,$$

where we use the fact that W is a subspace of V. Hence W is $g(T)$-invariant for any polynomial $g(t)$. □

5.4.11(a). Let $w \in W$. Since W is the T-cyclic subspace generated by v, the vector w can be written as $w = \sum_{i=0}^n a_i T^i(v)$ for some scalars $a_0, \cdots, a_n$. Then $T(w) = T(\sum_{i=0}^n a_i T^i(v)) = \sum_{i=0}^n a_i T(T^i(v)) = \sum_{i=0}^n a_i T^{i+1}(v) \in W$. Hence W is T-invariant. □

5.4.11(b). Let U be a T-invariant subspace of V containing v. Since U is T-invariant, $T(v), T^2(v), \cdots, T^k(v), \cdots \in U$. This implies that $\{v, T(v), \cdots, T^k(v), \cdots\} \subseteq U$. Hence $W = \text{span}\left(\{v, T(v), \cdots, T^k(v), \cdots\}\right) \subseteq U$. □

5.4.17. Let $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be the characteristic polynomial of A, then, by Cayley-Hamilton Theorem,

$$f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_0 I = 0.$$

This implies that A^n is a linear combination of $I, A, A^2, \cdots, A^{n-1}$. Multiplying A to the above equation from both sides, we have

$$Af(A) = A\left((-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_0 I\right) = (-1)^n A^{n+1} + a_{n-1}A^n + \cdots + a_0 A = 0.$$

This implies that A^{n+1} is a linear combination of $A, A^2, \cdots, A^n$ and, hence, a linear combination of $I, A, A^2, \cdots, A^{n-1}$. Inductively, for all positive integer $k \geq n$, A^k is a linear combination of $I, A, A^2, \cdots, A^{n-1}$. Therefore,

$$\dim\left(\text{span}(\{I, A, A^2 \cdots\})\right) = \dim\left(\text{span}(\{I, A, A^2 \cdots A^{n-1}\})\right) \leq n.$$

□

5.4.19. We proceed by induction. If $k = 1$, then the characteristic polynomial of A is $-(a_0 + t)$. The results hold. Assume the result holds for $k = n - 1$. For $k = n$, we compute the determinant by expanding the matrix $A - tI$ along the first row

$$\begin{aligned}
\det(A-tI) &= \det\begin{pmatrix} -t & 0 & \cdots & -a_0 \\ 1 & -t & \cdots & -a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{n-1} \end{pmatrix} \\
&= (-t)\det\begin{pmatrix} -t & 0 & \cdots & -a_1 \\ 1 & -t & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{n-1} \end{pmatrix} + (-1)^{n+1}(-a_0)\det\begin{pmatrix} 1 & -t & \cdots & 0 \\ 0 & 1 & \cdots & -a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \\
&= -t[(-1)^{n-1}(a_1 + a_2 t + \cdots + a_{n-1}t^{n-2} + t^{n-1})] + (-1)^n a_0 \\
&= (-1)^n(a_0 + a_1 t + \cdots + a_{n-1}^{n-1} + t^n).
\end{aligned}$$

□

5.4.42. Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in M_{n\times n}(\mathbb{R})$, then it is easy to see $\text{rank}(A) = 1$ and, hence by Dimension Theorem, $\text{nullity}(A) = n - 1$. Also by computing the determinant of $A - tI$, we know that

$$\begin{aligned}
\det(A-tI) &= \det\begin{pmatrix} 1-t & 1 & \cdots & 1 \\ 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-t \end{pmatrix} \\
&= \det\begin{pmatrix} n-t & 1 & \cdots & 1 \\ n-t & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n-t & 1 & \cdots & 1 \end{pmatrix} \\
&= (n-t)\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-t \end{pmatrix}.
\end{aligned}$$

By these results, we conclude that the characteristic polynomial of A is

$$(-1)^n t^{n-1}(t-n).$$

□