6.2.6. By Theorem 6.6, x can be uniquely written as $x = u+v$, where $u \in W$ and $v \in W^\perp$. Since $x \notin W$, we have $v \neq 0$. Let $y = v$, then

$$< x, y >=< u+v, v >=< u, v > + < v, v >= ||v||^2 > 0.$$

□

6.2.7. $\Rightarrow$ Since β is a basis for W. If $v \in \beta$, then $v \in W$. So for $z \in W^\perp$, we have $< z, v >= 0$, for all $v \in \beta$.
$\Leftarrow$ Since β is a basis for W. For all $u \in W$, $u = \sum_{i=1}^n a_i v_i$, where $a_1, \cdots, a_n$ are scalars and $v_1, \cdots, v_n \in \beta$. Hence, by assumption, we have

$$< z, \sum_{i=1}^n a_i v_i >= \sum_{i=1}^n \overline{a_i} < z, v_i >= 0.$$

this implies that $z \in W^\perp$. □

6.2.8. We proceed by induction on n. For $n = 1$, the statement $v_1 = w_1$ holds by the Gram-Schmidt process. Suppose that the statement holds for $n = k-1$. Now consider the orthogonal set of nonzero vectors $\{w_1, \cdots, w_n\}$. By induction hypothesis, we know that $v_1 = w_1, \cdots, v_{k-1} = w_{k-1}$. Then the Gram-Schmidt process says that

$$v_k = w_k - \sum_{i=1}^{k-1} \frac{< w_k, v_i >}{||v_i||^2} v_i = w_k - 0 = w_k.$$

This completes the proof. □

6.2.11. $\Rightarrow$ For $i, j = 1, 2, \cdots, n$,

$$(AA^*)_{ij} =< v_i, v_j >$$

where v_i is the i-th row vector of A. If $AA^* = I$, then

$$< v_i, v_j >= \delta_{ij}, \qquad i, j = 1, \cdots, n.$$

Hence $\{v_1, \cdots, v_n\}$ forms an orthonormal basis for $\mathcal{C}^n$.
$\Leftarrow$ For $i = 1, \cdots, n$, let v_i be the i-th row vector of A such that $\{v_1, \cdots, v_n\}$ forms an orthonormal basis for $\mathcal{C}^n$. Then $(AA^*)_{ij} =< v_i, v_j >= \delta_{ij}$ for $i, j = 1, \cdots, n$. This implies that $AA^* = I$. □

6.2.13(a). If $v \in S^\perp$, then v is orthogonal to all vectors in S. Since $S_0 \subseteq S$, we have v is orthogonal to all vectors in S_0. Hence $v \in S_0^\perp$. □

6.2.13(b). If $v \in S$, then v is orthogonal to all vectors in $S^\perp$. This means that $v \in (S^\perp)^\perp$, i.e. $S \subseteq (S^\perp)^\perp$. Since we know that $(S^\perp)^\perp$ is a subspace, we have $\text{span}(S) \subseteq (S^\perp)^\perp$. □

6.2.13(c). By (b), we have $W \subseteq (W^{\perp})^{\perp}$. By 6.2.6 we know that, if $x \notin W$, then there exists $y \in W^{\perp}$ such that $< x, y > \neq 0$, i.e. $x \notin (W^{\perp})^{\perp}$. This implies that if $x \in (W^{\perp})^{\perp}$, then $x \in W$ and, hence, $(W^{\perp})^{\perp} \subseteq W$. Therefore $W = (W^{\perp})^{\perp}$. □

6.2.13(d). By Theorem 6.6, we know that $V = W + W^{\perp}$. If $v \in W \cap W^{\perp}$, then $< v, v >= ||v||^2 = 0$. Hence $v = 0$. By combining these facts, we conclude that $V = W \oplus W^{\perp}$. □

6.2.15(a). By Theorem 6.5, we have

$$x = \sum_{i=1}^{n} < x, v_i > v_i, \qquad y = \sum_{j=1}^{n} < y, v_j > v_j.$$

Hence we have

$$\begin{aligned} < x, y >= & \langle \sum_{i=1}^{n} < x, v_i > v_i, \sum_{j=1}^{n} < y, v_j > v_j \rangle \\ = & \sum_{i=1}^{n} \sum_{j=1}^{n} < x, v_i > \overline{< y, v_j >} < v_i, v_j > \\ = & \sum_{i=1}^{n} < x, v_i > \overline{< y, v_i >}. \end{aligned}$$

□

6.2.15(b). Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis for V. Then

$$\phi_\beta(x) = [x]_\beta = \begin{pmatrix} < x, v_1 > \\ < x, v_2 > \\ \vdots \\ < x, v_n > \end{pmatrix}, \qquad \phi_\beta(y) = [y]_\beta = \begin{pmatrix} < y, v_1 > \\ < y, v_2 > \\ \vdots \\ < y, v_n > \end{pmatrix},$$

where $\phi_\beta : V \to F^n$ is the standard representation of V with respect to the basis β defined by $\phi_\beta(x) = [x]_\beta$. By combing these equalities with (a), we conclude that $\langle \phi_\beta(x), \phi_\beta(y) \rangle' = \langle [x]_\beta, [y]_\beta \rangle' =< x, y >$. □

6.2.18. Let $f \in W_o$ be an odd function, then, for any even function $g \in W_e$, we have

$$fg(x) = f(x)g(x) = -f(-x)g(-x) = -fg(-x),$$

i.e. fg is an odd function. Since fg is an odd function, then

$$< f, g >= \int_{-1}^{1} f(t)g(t)dt = 0.$$

Hence $W_o \subseteq W_e^{\perp}$.

Let f be an arbitrary function, then f can be written as $f = g + h$, where

$$g(x) = \frac{1}{2}(f(x) + f(-x)) \quad \text{and} \quad h(x) = \frac{1}{2}(f(x) - f(-x)).$$

It is easy to see that $g(x) = g(-x)$ is an even function and $-h(x) = h(-x)$ is an odd function. If $f \in W_e^{\perp}$, then we have

$$0 =< f, g >=< g + h, g >=< g, g > + < h, g >= ||g||^2,$$

since $< h, g >= 0$. So $g = 0$ and $f = h \in W_o$ is an odd function. Hence $W_e^{\perp} \subseteq W_o$. Therefore $W_e^{\perp} = W_o$. $\square$