NAME:____________ ID NO.:____________________ CLASS: __________

Problem 1:(10 points) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$,

(1) Determine all the eigenvalues of A.
(2) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
(3) Find a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.
(4) Determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. The eigenvalues are 4 and -1, a basis of eigenvectors is $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, $Q = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$, and $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$. □

Problem 2:(10 points)

(1) Prove that similar matrices have the same characteristic polynomial.

Proof. Assume that the $n \times n$ matrix A is similar to the $n \times n$ matrix B, then there exists an invertible $n \times n$ matrix Q such that $B = Q^{-1}AQ$. Now

$$\begin{aligned} \det(B - \lambda I) &= \det(Q^{-1}AQ - \lambda I) \\ &= \det(Q^{-1}(A - \lambda I)Q) \\ &= \det(Q^{-1}) \det(A - \lambda I) \det(Q) \\ &= \det(A - \lambda I). \end{aligned}$$

Hence the similar matrices A and B have the same characteristic polynomial. □

(2) Show that the definition of the characteristic polynomial of a liner operator on a finite-dimensional vector space V is independent of the choice of basis for V.

Proof. Let T be a liner operator on a finite-dimensional vector space V and let α and β be two ordered bases for V. Then there exists an invertible matrix Q such that

$$[T]_\alpha = Q^{-1}[T]_\beta Q,$$

where $[T]_\alpha$ and $[T]_\beta$ are matrix representations of T with respect to the ordered bases α and β, respectively. By (1), then $[T]_\alpha$ and $[T]_\beta$ have the same characteristic polynomial. Hence the definition of the characteristic polynomial of a liner operator on a finite-dimensional vector space V is independent of the choice of basis for V. □