NAME:____________ ID NO.:______________________ CLASS: __________

Problem 1:(10 points) Let $V = P_2(\mathbb{R})$ and T is defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.

(1) Test T for diagonalizability.
(2) If T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

Solution. Diagonalizable. Let γ be the standard ordered basis for $P_2(\mathbb{R})$, then $[T]_\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$,
and $[T]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\beta = \{1 - x^2, 1 + x^2, x\}$. □

Problem 2:(10 points)

(1) Suppose that $A \in M_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Proof. Since $\dim E_{\lambda_2} \geq 1$, we can choose a nonzero vector $v \in E_{\lambda_2}$. Let $\beta = \{v_1, v_2, \cdots, v_{n-1}\}$ be a basis for E_{λ_1}. Then, by Theorem 5.8, $\{v, v_1, v_2, \cdots, v_{n-1}\}$ forms a basis for F^n consisting of eigenvectors of A. Hence, Theorem 5.1 implies that A is diagonalizable. □

(2) Let $A \in M_{n\times n}(F)$. Then A and A^t share the same eigenvalues with the same multiplicities. Show by way of example that for a given common eigenvalue of A and A^t, these two eigenspaces need not be the same.

Proof. The matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ has eigenvalues 0 and 1. For the eigenvalue 0, $E_0 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is the eigenspace for A and $E_0' = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ is the eigenspace for A^t. □