Name:____________ Id No.:_____________________ Class: __________

Problem 1: Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^2 = A$.

(1) (5 points) Show that A is diagonalizable.

Proof. Let $g(t) = t^2 - t = t(t-1)$. Then $g(A) = O$, and hence the minimal polynomial $p(t)$ of A divides $g(t)$. This also implies that A has only possible eigenvalues 0 or 1. Since $g(t)$ has no repeated factors, neither does $p(t)$. Thus A is diagonalizable by Theorem 7.16. □

(2) (5 points) Show that $\operatorname{trace}(A) = \operatorname{rank}(A)$.

Proof.

$$\exists\, Q \in M_{n\times n}(\mathbb{R}),\ \ni\ Q^{-1}AQ = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} = D.$$

Hence $\operatorname{trace}(A) = \operatorname{trace}(QDQ^{-1}) = \operatorname{trace}(D) = \operatorname{rank}(A)$. □

(3) (5 points) Let $B \in M_{n\times n}(\mathbb{R})$ be another square matrix satisfying $B^2 = B$ such that $\operatorname{trace}(A) = \operatorname{trace}(B)$. Show that A is similar to B.

Proof.

$$\exists\, Q,\, P \in M_{n\times n}(\mathbb{R}),\ \ni\ Q^{-1}AQ = P^{-1}BP = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

This implies $A = (QP^{-1})^{-1}B(QP^{-1})$. Hence A is similar to B. □

Problem 2: (5 points) Find all possible 5×5 Jordan canonical forms which has the repeated eigenvalues -1 and 1, and the minimal polynomial $(t-1)^2(t+1)^2$.

Solution.

$$\begin{pmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & -1 & 1 \\ & & & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & -1 & 1 & \\ & & 0 & -1 & \\ & & & & -1 \end{pmatrix}.$$

□