NCU PHD PROGRAM ENTRANCE EXAM: ANALYSIS

(May 20, 2011)

Stage Setting: In the following problems, the functions are assumed be real-valued.

(1) (10%) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$.

(a) Prove that $\liminf a_n + \liminf b_n \le \liminf(a_n + b_n)$.

(b) If $\{a_n\}$ is convergent in $\mathbb{R}$, prove that $\liminf a_n + \liminf b_n = \liminf(a_n + b_n)$.

(2) (10%) Let $f(x, y) = \frac{xy}{(x^2+y^2)^2}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.

(a) Is f differentiable at $(0, 0)$? Give your reason.

(b) Is f Lebesgue integrable on $[-1, 1] \times [-1, 1]$? Give your proof.

(3) (15%) Assume f is Lebesgue integrable on $\mathbb{R}$.

(a) Prove that $g(y) \equiv \int_{-\infty}^{\infty} f(x)e^{-(x^2y^2)}dx$ is a bounded, continuous function on $\mathbb{R}$.

(b) If f is a continuous function on $\mathbb{R}$ and $f = 0$ almost everywhere with respect to Lebesgue measure, prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(4) (20%) Let $f, f_1, f_2, \cdots$ be Lebesgue integrable functions on $\mathbb{R}$, and, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $\mathbb{R}$.

(a) Prove or disprove that $f_n \to f$ in measure (Lebesgue measure).

(b) Prove or disprove that $\int_{\mathbb{R}} f dm = \lim\limits_{n\to\infty} \int_{\mathbb{R}} f_n dm$, where m is the Lebesgue measure.

(c) If f_n is absolutely continuous on $[0, 1]$ for each $n = 1, 2, \cdots$. Prove or disprove that f is absolutely continuous on $[0, 1]$.

(5) (15%) Let $(X, \mathcal{B})$ and (Y, Σ) be two measurable spaces and μ a measure on $\mathcal{B}$. Assume that $T : X \to Y$ has the property that $T^{-1}(A) \in \mathcal{B}$ for each $A \in \Sigma$. Let ν be a measure on Σ defined by $\nu(A) = \mu(T^{-1}(A))$ for each $A \in \Sigma$.

(a) If $f \in L^1(\nu)$, then show that $f \circ T \in L^1(\mu)$ and $\int_Y f d\nu = \int_X f \circ T d\mu$.

(b) If μ is finite and ω is a σ-finite measure on Σ such that $\nu \ll \omega$, then show that there exists a function $g \in L^1(\omega)$ such that $\int_X f \circ T d\mu = \int_Y f g d\omega$ holds for each $f \in L^1(\nu)$.

(6) (15%)

(a) Prove or disprove that $L^p([0, 1]) \supseteq L^q([0, 1])$, where $1 \le p < q < \infty$.

(b) Prove or disprove that $\ell^q \supseteq \ell^p$, where $1 \le p < q < \infty$, $\ell^p \equiv \{(x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ and $\ell^q \equiv \{(x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^q < \infty\}$.

(7) (15%) Let $\ell^2 = \{(a_1, a_2, \ldots) : \sum\limits_{n=1}^{\infty} |a_n|^2 < \infty\}$ be the normed space with the norm $\|(a_1, a_2, \ldots)\|_2 = (\sum\limits_{n=1}^{\infty} |a_n|^2)^{1/2}$.

(a) Let $U = \{(a_1, a_2, \ldots) \in \ell^2 : \|(a_1, a_2, \ldots)\|_2 \le 1\}$. Determine whether U is compact in $(\ell^2, \|\cdot\|_2)$, prove your answer.

(b) Let $A = \{(a_1, a_2, \ldots) \in \ell^2 : |a_n| \le \frac{1}{n}$ for all $n\}$. Determine whether A is compact in $(\ell^2, \|\cdot\|_2)$, prove your answer.