國立中央大學數學系一百學年度博士班入學考試

科目：代數

日期: 100 年 5 月 20 日

1. Let G be a finite group of order $p_1p_2p_3$ where $p_i, i = 1, 2, 3$ are three (not necessarily distinct) prime numbers.

   (a) (% 15) Suppose that $p_1 < p_2 < p_3$ with $p_1p_2 \not\equiv 1 \pmod{p_3}$. Show that there exists a subgroup H of G of order p_2p_3. How many subgroup of G is of order p_2p_3? Is it true that every p_2-Sylow subgroup of G is also a p_2-Sylow subgroup of H? (You need to explain your answers.)

   (b) (% 10) Suppose $p = p_1 < p_2 = p_3 = q$. Fix a p-Sylow subgroup P and a q-Sylow subgroup Q of G. Prove or disprove the following statement:
   The group G is abelian if and only if $xy = yx$ for $x \in P$ and $y \in Q$.

2. Let $R = k[t]$ be a polynomial ring in variable t over the field k and let M be a finitely generated R-module.

   (a) (% 10) Note that M is equipped with a k-vector space stucture. Show that M is a torsion R-module if and only if M is a finite dimensional k-vector space. (A module is called a torsion module if every nonzero element $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$.)

   (b) (% 10) Suppose that for every exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules, the following sequence

   $$0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'') \to 0$$

   is also exact. Prove that M is a free R-module.

3. (% 10) Let $f(X) = u(X - t_1) \cdots (X - t_n)$. The discriminant of f is defined by

   $$D(f) = (-1)^{n(n-1)/2} u^{2n-2} \prod_{i \neq j} (t_i - t_j).$$

   Let $n \geq 2$ be an integer and let the polynomial $f(x) = x^n + ax + b$ where a, b are the coefficients of f. Compute the discriminant $D(f)$ of f in terms of a, b and n.

4. Let p be a prime number and let $a \in \mathbb{Z}$ be a non-zero integer which is not a p-th power. Let F be a splitting field of the polynomial $f(x) = x^p - a$ over $\mathbb{Q}$.

   (a) (% 10) Prove that $\mathrm{Gal}(F/\mathbb{Q})$ has a unique p-Sylow subgroup $\mathcal{P}$.

(b) (% 10) Compute the fixed field $F^{\mathcal{P}}$ of $\mathcal{P}$.

5. Let $\mathbb{Z}[x]$ denote the polynomial ring in the variable x over the ring of integers $\mathbb{Z}$.

   (a) (% 10) Is $\mathbb{Z}[x]$ a principal ideal domain (p.i.d)? If your answer is yes, give a proof; otherwise, give an ideal of $\mathbb{Z}[x]$ and show that it is not a principal ideal.

   (b) (% 15) Determine all prime ideals of $\mathbb{Z}[x]$. Let $\mathfrak{M}$ be a maximal ideal of $\mathbb{Z}[x]$. Is it true that $\mathbb{Z}[x]/\mathfrak{M}$ is a finite field? Explanation of your answers is required.