NCU PHD PROGRAM ENTRANCE EXAM: ANALYSIS

(May 18, 2012)

Stage Setting: In the following problems, the functions are assumed be real-valued.

1. (15%) Let f be a Lebesgue integrable function on $(0,\infty)$. Prove that the function

$$g(t) = \int_{(0,\infty)} e^{-tx} f(x)dx, \quad 0 < t < \infty,$$

is bounded, differentiable and $g'(t) = -\int_{(0,\infty)} xe^{-tx} f(x)dx$.

2. (10%) Let $f, f_1, f_2, \cdots$ be measurable functions on a finite measure space $(X, \mathfrak{B}, \mu)$. Prove that $f_n \to f$ in measure if and only if

$$\int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu \to 0 \text{ as } n \to \infty.$$

3. (10%) Let μ be the Lebesgue measure on $[0,1]$, ν be the counting measure on $[0,1]$ and

$$\triangle = \{(x,x) : x \in [0,1]\}.$$

Determine whether the characteristic function $\chi_\triangle$ is $\mu \times \nu$-integrable on $[0,1] \times [0,1]$. Justify your answer.

4. (10%) Let $(X, \mathcal{B}, \mu)$ be a finite measure space, $f : X \to \mathbb{R}$ be a measurable function and $\alpha \in \mathbb{R}$. If f^n is integrable and $\int_X f^n d\mu = \alpha$ for all $n = 1, 2, \ldots$, prove that $f = \chi_E$ for some measurable subset E of X.

5. (15%) Let μ and ν be finite measures on the measurable space $(X, \mathcal{B})$ with $\mu \ll \nu$ and $\nu \ll \mu$.

   (a) Let f be a nonnegative measurable function on X. Prove that $\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu$.

   (b) Find a relation of $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$. Prove your assertion.

6. (10%) If two functions $f, g \in L^3(X, \mathcal{B}, \mu)$ satisfy

$$\|f\|_3 = \|g\|_3 = \int_X f^2 g d\mu = 1,$$

then show that $g = |f|$ almost everywhere holds.

7. (10%) Determine whether the set $C[0,1] \equiv \{f : [0,1] \to \mathbb{R} : f \text{ is continuous on } [0,1]\}$ with the metric $d(f,g) \equiv \int_0^1 |f(x) - g(x)|dx$ is a complete metric space. Give your reasons.

8. (10%) Is $(l^\infty, \|\cdot\|_\infty)$ separable? Justify your answer. (Recall that a metric space X is separable if X has a countable dense subset, and $l^\infty = \{(a_1, a_2, \ldots, a_n, \ldots) : \sup_n |a_n| < \infty\}$.)

9. (10%) Let $f \in C^\infty([0,1])$. Prove that for any $n \in \mathbb{N}$ there exists a sequence of polynomials $\{p_m\}$ such that $\sum_{k=0}^n \|p_m^{(k)} - f^{(k)}\|_\infty \to 0$ as $m \to \infty$.