中央大學數學系博士生入學考：分析 (May. 9, 2013)

1. (15%) Suppose f is a measurable function on $\mathbb{R}$ and B is a Borel subset of $\mathbb{R}$. Prove $f^{-1}(B)$ is a measurable set.

2. (15%) Let $T(x) := x^2$ on $\mathbb{R}$. Does T map sets of measure zero into sets of measure zero? Give your reason.

3. (15%) Let

$$f(x) = \begin{cases} x\sin(1/x) & \text{for } 0 < x \le 1, \\ 0 & \text{for } x = 0. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^3\cos(1/x) & \text{for } 0 < x \le 1, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Determine whether f and g are of bounded variation on $[0,1]$. Give your proof in each case.

(b) Determine whether f and g are absolutely continuous functions on $[0,1]$. Give your proof in each case.

4. (15%) If f and g are real-valued functions on $E \subset \mathbb{R}^n$ with $f, g \in L^3(E)$ satisfy

$$\|f\|_3 = \|g\|_3 = \int_E f^2 g = 1,$$

then show that $g = |f|$ almost everywhere on E.

5. (15%) Let f and $\{f_n\}$ be measurable functions which are defined and finite almost everywhere in a set E with $|E| < \infty$. Suppose $\{f_n\}$ converges in measure on E to f. Dose f_n converge to f almost everywhere on E? Give your reason.

6. (15%) If $f \in L^1(\mathbb{R}^n)$. Show that there exists a sequence $\{C_k\}$ of continuous functions with compact support such that

$$\int_{\mathbb{R}^n} |f - C_k| dx \to 0 \quad \text{as } k \to \infty.$$

7. (15%) Let ϕ be a bounded measurable function on $\mathbb{R}^n$ such that $\phi(x) = 0$ for $|x| \ge 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $f \in L^1(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0} (f * \phi_\varepsilon)(x) = f(x) \qquad \text{in the Lebesgue set of } f.$$

Here $(f * g)(x)$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$