Numerical Analysis

2013 Ph.D. Program Entrance Exam for the Dept. of Math, Nat'l Central Univ.

Problem 1 (System of Linear Equations and Numerical Linear Algebra).

Complete the following.

1. (15%) Let A be a diagonally dominant matrix. State the Gauss-Seidel iterative scheme to obtain the solution to $\mathrm{A}x = b$, and show that the Gauss-Seidel iteration converges for any starting vector.

2. (10%) Let A be an $n \times n$, symmetric and positive definite matrix. Show that the solution to $\mathrm{A}x = b$ is the minimizer of the quadratic form

$$q(x) = \langle x, \mathrm{A}x \rangle - 2\langle x, b \rangle \,,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$.

Problem 2 (Approximating Functions).

(15%) Suppose that $f : [0,1] \to \mathbb{R}$ is known to be bounded, and continuous at a point $a \in (0,1)$, and p_n be the Bernstein polynomial of degree n associate to f given by

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k} \,.$$

Show that $p_n(a)$ converges to $f(a)$. **Note that f might be continuous at a only.**

Problem 3 (Numerical Differentiation and Integration).

Let $x_i = \dfrac{i}{\mathrm{n}}$. Define two matrices $\mathrm{M}, \mathrm{D} : \mathbb{R}^{\mathrm{n}+1} \to \mathbb{R}^{\mathrm{n}+1}$ by

$$\mathrm{D} = [d_{ij}] \qquad \text{with} \qquad d_{ij} = \int_0^1 \varphi_i(x) \frac{d\varphi_j}{dx}(x) dx \,,$$

where $\{\varphi_i\}_{i=1}^{\mathrm{n}+1}$ are piecewise linear functions satisfying

1. $\varphi_i(x_{i-1}) = 1$;

2. $\varphi_i(x_k) = 0$ for all $k = 1, 2, \cdots, n$ except $k = i-1$.

3. φ_i is linear in between $[x_{k-1}, x_k]$ for all $k = 1, \cdots, n$.

Complete the following.

1. (10%) Use the two-point Gaussian quadrature method; that is,

$$\int_{-1}^{1} f(x) dx \approx f\big(-\frac{1}{\sqrt{3}}\big) + f\big(\frac{1}{\sqrt{3}}\big) \,,$$

to find the **exact value** of d_{ij}. Explain why the Gaussian quadrature method can be used to obtain the exact value of the integral.

2. (10%) Explain why D is an approximation of the differential operator in the sense that if $f : [0,1] \to \mathbb{R}$ is a function, and vector $\mathrm{F} = \big(f(x_0), f(x_1), \cdots, f(x_n)\big)^{\mathrm{T}}$, then $(\mathrm{D}F)_i$ is approximately $f'(x_i)$ for all i. Find the order of this approximation.

Problem 4 (Numerical ODE).

(15%) Show that

$$y_{n+1} - y_n = hf\big(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n + \frac{h}{2}, y_n + \frac{h}{3} f(x_n, y_n))\big)$$

or equivalently,

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf\big(x_n + \frac{h}{2}, y_n + \frac{k_1}{3}\big) \\
k_3 &= hf\big(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\big) \\
y_{n+1} &= y_n + k_3
\end{aligned}$$

is a third order method of solving the ODE

$$y' = f(x, y)\,.$$

Problem 5 (Numerical PDE & Newton's method).

In numerical analysis, the **Crank-Nicolson method** is a finite difference method used for numerically solving partial differential equations of the form

$$\frac{\partial u}{\partial t} = F\big(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\big)\,,$$

and the Crank-Nicolson method suggests the following second order scheme:

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} = \frac{1}{2}\Big[F_k^{n+1}\big(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x}\big) + F_k^{n+1}\big(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x}\big)\Big]\,,$$

where $u_k^n \equiv u(x_k, n\Delta t)$, and the function F must be discretized spatially with a central difference.

Consider the (nonlinear) Partial Differential Equation

$$\begin{aligned}
u_t(x,t) + uu_x - u_{xx}(x,t) &= 0 && \forall\, x \in [0,1]\,, t > 0\,, \\
u(x,0) &= u_0(x) && \forall\, x \in [0,1]\,, \\
u(0,t) = u(1,t) &= 0 && \forall\, t > 0\,.
\end{aligned}$$

Let $\big\{0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1\big\}$ be a uniform partition of the domain $[0,1]$, and the time step be $\Delta t \equiv \Delta x^2$.

1. (10%) Let $u^n \equiv (u_1^n, \cdots, u_{N-1}^n)^{\mathrm{T}}$. Using the Crank-Nicolson method, one obtains that u^n satisfies the relation

$$G(u^{n+1}) \equiv (\mathrm{I} - \mathrm{A})u^{n+1} + f(u^{n+1}) = \mathrm{A}u^n - f(u^n)\,, \qquad (\star)$$

for some matrix A and nonlinear function f, where I is the identity matrix. Find A and f.

2. (15%) Write down the iterative scheme to solve the nonlinear equation $(\star)$ using Newton's method.