1. **(a)** Prove that no group of order 48 is simple. (12%)

   (b) Classify all the groups of order 49 (up to isomorphism). (8%)

2. Let H, K be two proper subgroups of G such that both indices $[G:H], [G:K]$ are finite. Show that

   (a) $[G:H\cap K] \leq [G:H][G:K]$ (10%)

   (b) $[G:H\cap K] = [G:H][G:K]$ if and only if $G = HK$. (15%)

3. Let p be a prime number. Show that there is "no" rational solution $x \in \mathbb{Q}$ satisfying

$$x^5 + px + p = 0. \qquad (10\%)$$

4. Let m, n be positive integers. Determine the abelian group $\mathbb{Z}_m \bigotimes_{\mathbb{Z}} \mathbb{Z}_n$ (up to isomorphism) where $\mathbb{Z}$ is the group of integers. (10%)

5. Find an irreducible polynomial $p(x)$ in $\mathbb{Q}[x]$ so that $\mathbb{Q}(\sqrt{1+\sqrt{5}})$ is isomorphic to $\mathbb{Q}[x]/ < p(x) >$. You need to verify that $p(x)$ is irreducible over $\mathbb{Q}$ precisely. (15%)

6. Let F be a field of characteristic zero, and let $F \leq E \leq K \leq \overline{F}$, where E is a normal extension of F and K is an extension of F by radicals and $\overline{F}$ is an algebraic closure of F. Prove that the Galois group $G(E/F)$ is a solvable group. (20%)