In this test X always denotes the closed unit interval $[0, 1]$ in the real line $\mathbb{R}$.

1. State Lebesgue's dominated convergence theorem. Please clearly explain the notations you will use.

2. State Fatou's lemma. Please clearly explain the notations you will use.

3. Let $[0, 1]$ be the unit interval in the real line. What are the dual spaces of the following Banach spaces?

   - $L^1(X)$.
   - $L^4(X)$.
   - $C(X)$, the space of continuous functions on $[0, 1]$.

4. (1) State the definition of a compact space.

   (2) Let $f : X \to \mathbb{R}$ be a continuous function from $[0, 1]$ into $\mathbb{R}$. Show that the image $f(X)$ is compact.

   (3) Show that the image $f(X)$ in part (2) is indeed a closed interval.

5. (a) State the definition of uniform convergence. (State it only for functions on the interval $[0, 1]$.)

   (b) Show that if $f_n(x), n = 1, 2, \cdots$, is a sequence of continuous functions on $[0, 1]$, and $f_n(x)$ converges uniformly to a function $f(x)$ on $[0, 1]$, then $f(x)$ is continuous.

6. (a) State the definition of uniform continuity. (State it only for functions on the interval $[0, 1]$.)

   (b) Show that any continuous function on $X = [0, 1]$ is also uniformly continuous.

7. Use the Holder inequality to determine which Lebesgue space is bigger: $L^2(X)$ or $L^3(X)$?