國立中央大學九十四學年度數學系博士班招生筆試試題卷

歸檔

科目：代數

1. (25%) Let p, q be prime numbers and let G be a group of order p^2q. Prove or disprove that G is solvable. (Hint: Consider the cases $p < q$, $p = q$, $p > q$.)

2. (25%)
   (a) Determine the ring of algebraic integers in $\mathbb{Q}(\sqrt{-3})$ and show that it is a principal ideal domain.
   (b) Let R be a principal ideal domain. Show that a finitely generated R-module is free if and only if it is torsion-free.
   (c) Let R be a principal ideal domain and M a finitely generated R-module. Show that the following sequence is split exact:
   $$0 \longrightarrow M_{\text{tor}} \longrightarrow M \longrightarrow M/M_{\text{tor}} \longrightarrow o$$
   where M_{tor} is the torsion submodule of M.

3. (25%)
   (a) Let R be a local ring, m its maximal ideal, $k = R/m$ its residue field. Let M be a finitely generated R-module. Show that $\{x_1, \ldots, x_n\}$ is a minimal generating set of M if and only if the images $\bar{x_1}, \ldots, \bar{x_n}$ in M/mM form a basis for M/mM over k.
   (b) Let R be a local ring. Show that any finitely generated projective module over R is free.
   (c) Let R be a commutative ring and M a finitely generated R-module. Assume that $M \otimes_R \kappa(m) = 0$ for every maximal ideal m, where $\kappa(m)$ is the residue field of the local ring R_m. Show that $M = 0$.

4. (10%) Let K be the splitting field of $f(x) = x^5 + 32x^3 + x^2 + 31x + 31$ over the rational numbers $\mathbb{Q}$. Find the extension degree $[K : \mathbb{Q}]$ and the Galois group of K over $\mathbb{Q}$.

5. (15%) Let K be an infinite Galois extension over k with Galois group G. Prove that G is profinite. (A group is called profinite if it is an inverse limit of finite groups).