科目：

NCU PHD PROGRAM ENTRANCE EXAM: ANALYSIS

1. Prove that a bounded function $f : [0,1] \to \mathbb{R}$ is Riemann integrable if and only if f is continuous almost everywhere.

2. (a) Give the precise definition of $Df(p)$ of a mapping $f : U \to \mathbb{R}^m$, where U is an open subset in $\mathbb{R}^n$ and $p \in U$, and state the inverse function theorem on $\mathbb{R}^n$.

   (b) Prove the inverse function theorem in the following special situation: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1, $f(0) = 0$ and $Df(0) = I_{n\times n}$. For any $y \in \bar{B}_r(0)$ we define $g_y(x) = y + (x - f(x))$. Show that if r is chosen small enough than g_y maps $\bar{B}_{2r}(0)$ into $\bar{B}_{2r}(0)$ and there is a unique fixed point $x \in \bar{B}_{2r}(0)$ of g_y, which is equivalent to that $f(x) = y$.

   (c) Part (b) gives a well-defined inverse mapping $f^{-1} : \bar{B}_r(0) \to \bar{B}_{2r}(0)$. Prove that f^{-1} is continuous.

   (d) Prove that f is C^1 on $B_r(0)$ and give the formula for $D(f^{-1})(y)$.

3. Prove that $L^2 = L^2([-\pi,\pi])$ is a Hilbert space. Moreover prove that the trigonometric functions form a Hilbert basis.

4. (a) Let f be a nonnegative function which is integrable over a measurable set E. Prove that for any given $\epsilon > 0$ there is a $\delta > 0$ such that for every measurable subset $A \subset E$ with $m(A) < \delta$ we have

$$\int_A f < \epsilon.$$

   (b) Let f be a Lebesgue integrable function on $[a,b]$ and

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

   Prove that F is absolutely continuous on $[a,b]$ and $F'(x) = f(x)$ for almost all x in $[a,b]$.

5. (a) Let $K \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} K = 1$ and let $K_\epsilon(x) = \epsilon^{-n}K(\frac{x}{\epsilon})$. Define

$$f_\epsilon(x) = (f * K_\epsilon)(x) = \int_{\mathbb{R}^n} f(t)K_\epsilon(x-t)\,dt, \quad x,t \in \mathbb{R}^n.$$

   Prove that if $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ then $\|f_\epsilon - f\|_p \to 0$ as $\epsilon \to 0$.

   (b) Prove that C_0^∞ is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.

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