國立中央大學九十二學年度碩士班考試入學招生試題卷

系所別: 數學系 科目: 數值分析

參考用

共五大題，每一大題20分，共計100分

1. The following table lists values of a function $y(t)$ and its derivative $y'(t)$ at various points t_i.

i	t_i	$y(t_i)$	$y'(t_i)$
0	-0.5	-0.02475	0.751
1	-0.25	0.3349375	2.189
2	0	1.101	4.002

   (a) Find the Lagrange interpolating polynomial $P_2(t)$.

   (b) Find the Hermite interpolating polynomial $H_5(t)$.

2. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ as follows:

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6, \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15. \end{aligned}$$

   (a) Formulate the Gauss-Seidel iterative method with an initial approximation $\mathbf{x}^{(0)}$.

   (b) Explain why the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ generated by the Gauss-Seidel method in part (a) will converge to the unique solution $\mathbf{x}$.

3. Let $\mathbf{A} \in \mathbf{R}^{n\times n}$ be a given symmetric and positive definite matrix and $\mathbf{b} \in \mathbf{R}^n$ a given vector.

   (a) Show that matrix $\mathbf{A}$ has n real and positive eigenvalues.

   (b) Suppose that $\mathbf{x}^*$ is an approximation to the solution of $\mathbf{Ax} = \mathbf{b}$. Show that

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|_2}{\|\mathbf{x}\|_2} \le \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)\frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2}, \quad \text{provided } \mathbf{x} \ne \mathbf{0} \text{ and } \mathbf{b} \ne \mathbf{0},$$

   where $\mathbf{r}$ is the residual vector for $\mathbf{x}^*$; $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{A}$, respectively.

4. (a) Assume that $u \in C^4[x_0 - h, x_0 + h]$. Use Taylor's theorem to derive the following formulas for approximating $u'(x_0)$ and $u''(x_0)$:

$$\begin{aligned} u'(x_0) &= \frac{1}{h}\Big\{u(x_0) - u(x_0 - h)\Big\} + \frac{h}{2}u''(\xi), \quad \text{for some } \xi \in (x_0 - h, x_0); \\ u''(x_0) &= \frac{1}{h^2}\Big\{u(x_0 - h) - 2u(x_0) + u(x_0 + h)\Big\} - \frac{h^2}{12}u^{(4)}(\eta), \quad \text{for some } \eta \in (x_0 - h, x_0 + h). \end{aligned}$$

   (b) Consider the following two-point boundary value problem:

$$\begin{cases} -\varepsilon u''(x) + u'(x) + u(x) = 0, & 0 < x < 1, \\ u(0) = 1, \ u(1) = 0, \end{cases}$$

   where ε is a given positive constant. Let $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ be a uniform partition of $[0, 1]$ with mesh size $h > 0$. Applying the numerical differentiation formulas derived in part (a) to approximate the solution of the boundary value problem, we can obtain a linear system of the form $\mathbf{A}U = \mathbf{b}$, where $U = (U_1, U_2, \cdots, U_{N-1})^{\top}$, $U_0 = 1$, $U_N = 0$, and U_i denotes the approximation to $u(x_i)$. Find the $(N-1) \times (N-1)$ matrix $\mathbf{A}$ and $(N-1) \times 1$ vector $\mathbf{b}$.

5. Derive the Newton method with initial point $(x^{(0)}, y^{(0)}, z^{(0)})^{\top}$ for finding a root the following nonlinear system:

$$\begin{cases} f(x, y, z) &= 0, \\ g(x, y, z) &= 0, \\ h(x, y, z) &= 0. \end{cases}$$