所別：數學系碩士班 不分組 科目：抽象代數

◇ Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively.

1. (15 %) Let G be an abelian group which contains two finite cyclic subgroups H and K of order s and t respectively. Show that G contains a cyclic subgroup of order the least common multiple of s and t. (Hint: you can deal with the case that s and t are relatively prime first.)

2. (15 %)
   (a) Let H be a normal subgroup of G of order 2. Show that H is in the center of G.
   (b) Assume that G is a finite group and p is the smallest prime dividing $|G|$. Let H be a normal subgroup of order p in G. Show that H is in the center of G. (Hint: you can consider the conjugation action of G on the set $H \setminus \{e\}$ where e is the identity of G.)

3. (10 %) How many elements of order 17 are contained in a group of order 255?

4. (10 %) Show that if D is a UFD, then a finite product of primitive polynomials in $D[x]$ is again primitive.

5. (20 %) Prove or disprove the following.
   (a) The polynomial ring $\mathbb{R}[x, y]$ in two variables is a Euclidean domain.
   (b) The polynomial ring $\mathbb{R}[x]$ in one variable is a PID.

6. (10 %) Show that every finite extension field of $\mathbb{R}$ is either $\mathbb{R}$ itself or is isomorphic to $\mathbb{C}$.

7. (20 %) Let K be the splitting field of $x^5 - 1$ over $\mathbb{Q}$.
   (a) Describe the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
   (b) Determine all intermediate fields between K and $\mathbb{Q}$.