ALGEBRA

September 2003

1. (15%) Show that there are no non-abelian simple groups of order < 60.

2. (10%) Let $m \in \mathbf{Z}$ be square-free and write A for the integral closure of $\mathbf{Z}$ in $\mathbf{Q}[\sqrt{m}]$. Show that $A = \mathbf{Z}[(1+\sqrt{m})/2]$ if $m \equiv 1 \pmod 4$ and $A = \mathbf{Z}[\sqrt{m}]$ otherwise.

3. (10%) If R is Noetherian, then $R[[x]]$ is also Noetherian.

4. (15%)

(a) Let R be a commutative ring and I an ideal contained in every maximal ideal of R. Suppose that M is a finitely generated R-module and $IM = M$. Then $M = 0$.

(b) Let R be a local ring and M a finitely generated projective R-module. Then M is free.

5. (15%)

(a) For each positive integer n, there exists an extension L of a field K such that $\mathrm{Gal}(L/K) \simeq S_n$.

(b) Is every finite group isomorphic to some Galois group $\mathrm{Gal}(F/K)$ for some extension F of some field K? Justify your answer.

6. (15%) Are the following polynomial equations solvable by radicals over $\mathbf{Q}$? Explain your answers.

(a) $x^n - 1 = 0$, $n \geq 7$, $n \in \mathbf{N}$.

(b) $x^5 - 7x^2 + 7 = 0$.

7. (20%)

(a) Let L be a cyclic extension of a field K of degree n and $\mathrm{Gal}(L/K) = \langle \sigma \rangle$. For $\alpha \in L\backslash\{0\}$, show that the norm $N_{L/K}(\alpha) = 1$ if and only if there exists $\beta \in L\backslash\{0\}$ such that $\alpha = \beta/\sigma(\beta)$.

(b) Let $\mathrm{char} K \nmid n$ and K contain a primitive n-th root of unity. Show that if L is a cyclic extension of K of degree n, then $L = K(\alpha)$, where α satisfies $x^n - a = 0$ for some $a \in K$. Conversely, let $a \in K$. Show that if α is a root of $x^n - a$, then $K(\alpha)$ is a cyclic extension of K of degree d with $d|n$ and $\alpha^d \in K$.