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Algebra

Qualify Exam., Spring 2006

(25%) 1. Let p be a prime. A group is called a p-group if the order of each element in G is a power of p.

(a) Prove that a finite group is a p-group if and only if the order of G is a power of p.

(b) Prove that every finite p-group is solvable.

(c) Prove that every finite group of order 5625 is solvable.

(20%) 2. Let R be a commutative ring with identity and let I, J be ideals of R with $I + J = R$. An R-module M is said to be cyclic if $M = aR$ for some $a \in M$.

(a) Prove or disprove that the product $R/I \times R/J$ is a cyclic R-module.

(b) Prove or disprove that the tensor product $R/I \otimes R/J$ is a cyclic R-module.

(15%) 3. Prove or disprove that every finitely generated torsion free module over a unique factorization domain is free.

(20%) 4. Let n be a positive integer and let ζ_n be a primitive n-th root of unity. Let p be a prime and let $\mathbb{F}_p$ denote the finite field with p elements.

(a) Determine the extension degree $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$ and the separable extension degree $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]_s$.

(b) Determine the extension degree $[\mathbb{F}_p(\zeta_n) : \mathbb{F}_p]$ and the separable extension degree $[\mathbb{F}_p(\zeta_n) : \mathbb{F}_p]_s$.

(c) Suppose that $p \nmid n$. Denote $R = \mathbb{Z}[\zeta_n]$ and $I = pR$. Prove that I is a maximal ideal of R if and only if $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = [\mathbb{F}_p(\zeta_n) : \mathbb{F}_p]$.

(20%) 5. Denote K to be the splitting field of the polynomial $f(x) = x^5 + 9x + 9$ over $\mathbb{Q}$. Find the Galois group of K over $\mathbb{Q}$ and determine whether the equation $f(x) = 0$ is solvable by radicals. Explain why.