Algebra

February 12, 2001

1. Find counter examples and explain what is wrong in the proofs:

   (a) (5 pts.) Let $N_1 \subseteq N_2$ be two normal subgroups of G. Then we have that

   $$(G/N_2) \times (N_2/N_1) \cong G/N_1.$$

   proof. Since $N_2 \times N_1 \cong N_1 \times N_2$, it implies that $G/N_2 \times N_2/N_1 \cong (G \times N_2)/(N_2 \times N_1) \cong (G \times N_2)/(N_1 \times N_2) \cong G/N_1 \times N_2/N_2 \cong G/N_1$.

   (b) (5 pts.) Suppose that R is a finite ring of characteristic p, where p is a prime integer. Then $|R| \equiv 1 \pmod{p}$. ($|R|$ is the number of elements in R.)

   proof. Consider the group $\mathbb{Z}/p\mathbb{Z}$ acts on the set R by $\bar{n} \cdot r = nr$ for all $r \in R$, $n \in \mathbb{Z}$ ($\bar{n}$ means the residue of n modulo p). Consider the set $R_0 = \{r \in R \,|\, \bar{n} \cdot r = r, \forall \bar{n} \in \mathbb{Z}/p\mathbb{Z}\}$. Since p is a prime, it implies that $R_0 = \{0\}$. Therefore $|R| \equiv |R_0| = 1 \pmod{p}$ and our claim follows.

2. Let H be a normal subgroup of a Group G and let $\pi : G \to G/H$ be the canonical epimorphism (*i.e.* $\pi(g) = gH$).

   (a) (8 pts.) Suppose that P is a Sylow p-subgroup of G. Prove that $\pi(P)$ is a Sylow p-subgroup of G/H.

   (b) (8 pts.) Let $\wp$ be a Sylow p-subgroup of G/H. Show that there exists a Sylow p-subgroup P of G such that $\pi(P) = \wp$.

   (c) (9 pts.) Suppose that for any Sylow p-subgroup P of G, $hPh^{-1} = P$, $\forall\, h \in H$. Prove that π gives a one-to-one correspondence between the set of all Sylow p-subgroups of G and the set of all Sylow p-subgroups of G/H.

3. (a) (5 pts.) Let $\mathbb{R}$ be the field of real numbers. Prove that any polynomial $f(x) \in \mathbb{R}[x]$ can be factorized as a product of polynomials in $\mathbb{R}[x]$ of degree 1 or 2.

   (b) (10 pts.) Let K be a finite extension of degree n over a field k and suppose that $\text{char}(k) \nmid n$, if $\text{char}(k) \neq 0$. Prove that K is an algebraic closure of k if and only if every polynomial $f(x) \in k[x]$ can be factorized as a product of polynomials in $k[x]$ of degree dividing n.