Analysis Ph. D. Qualifying Examination (2004.02.12)

Note: In the following, all functions are real-valued!

1. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and f be a nonnegative measurable function on X. Prove that there exists a Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions in $L^1(X,\mu)$ such that $f_n \to f$ almost uniformly if and only if $\sup\{\int_X \varphi d\mu : \varphi \text{ is simple and } \varphi \le f\} < \infty$. (10%)

2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of numbers in $[0,1]$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{|x-a_n|}}$ is finite for almost all x in $[0,1]$. (10%)

3. Let μ and ν be σ-finite measures on $(X, \mathcal{B})$.

   (a) If $\nu \ll \mu$, prove that $\int_X f d\nu = \int_X f \cdot \frac{d\nu}{d\mu} d\mu$ for all ν-integrable functions f on X. (10%)

   (b) If $\nu \ll \mu$ and $\mu \ll \nu$, find a relation of $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$ and prove your assertion. (4%)

4. Let

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0; \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^3\sin(1/x) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Determine whether f and g are of bounded variation on $[0,1]$. Give your proof in each case. (10%)

   (b) Determine whether f and g are absolutely continuous on $[0,1]$. Give your proof in each case. (6%)

5. Let $f(x,y) = \frac{xy}{(x^2+y^2)^2}$ for $-1 \le x, y \le 1$. Determine whether any of $\int_{-1}^{1}(\int_{-1}^{1} f(x,y)dx)dy$, $\int_{-1}^{1}(\int_{-1}^{1} f(x,y)dy)dx$ and $\int_{[-1,1]\times[-1,1]} f(x,y)dxdy$ exists. For the existing ones, give their value; for the others, give your proof for their nonexistence. (10%)

6. Let f be a function in $L^p[0,1]$ $(1 \le p < \infty)$ and let $F(x) = \int_0^x f(t)dt$ for x in $[0,1]$.

   (a) Prove that $\|F\|_p \le \frac{1}{\sqrt[p]{p}}\|f\|_p$. (5%)

   (b) Give a necessary and sufficient condition on f only (not involving F) for which "=" holds in (a). (5%)

7. Let S be a linear subspace of $L^q[0,1]$ that is closed as a subspace of $L^p[0,1]$, where $1 < p < q < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in S. Prove that $\{f_n\}_{n=1}^{\infty}$ is convergent in $(L^q[0,1], \|\cdot\|_q)$ if and only if $\{f_n\}_{n=1}^{\infty}$ is convergent in $(L^p(X,\mu), \|\cdot\|_p)$. (10%)

8. Let $A = [a_{ij}]$ be an n-by-n matrix with $a_{ij} > 0$ for all i and j and $\sum_{j=1}^{n} a_{ij} = 1$ for all i. Prove that there is a unique vector $x_0 = [x_1, x_2, \cdots, x_n]^T$ with $x_i \ge 0$ for all i and $\sum_{i=1}^{n} x_i = 1$ such that $Ax_0 = x_0$. (10%)

9. Let f be a continuous function on $[0,1]$. Prove that

$$\lim_{n\to\infty} \frac{\int_0^1 x^n f(x)dx}{\int_0^1 x^n dx} = f(1). \quad (10\%)$$