中央大學博士班資格考〔分析〕

(2005.09)

Note: In the following, all functions are real-valued!

1. Assume f is Lebesgue integrable on $\mathbb{R}$. Prove that $g(y) \equiv \int_{-\infty}^{\infty} f(x)e^{-(x^2y^2)}dx$ is a bounded, continuous function on $\mathbb{R}$. (10%)

2. Let $f, f_1, f_2, \cdots$ be measurable functions on the measure space $(X, \mathcal{B}, \mu)$, and, $f_n \leq f_{n+1}$ for $n = 1, 2, \cdots$.

   (a) If $f_n \to f$ in measure, prove that $f_n \to f$ almost everywhere. (10%)

   (b) If $f_n \to f$ almost everywhere, prove or disprove that $f_n \to f$ in measure. (5%)

3. Let $f, f_1, f_2, \cdots$ be Lebesgue integrable functions on $[0,1]$, and, $\{f_n\}_{n=1}^{\infty}$ converge uniformly to f.

   (a) Prove or disprove that $\int_{[0,1]} f dm = \lim_{n\to\infty} \int_{[0,1]} f_n dm$, where m is the Lebesgue measure. (5%)

   (b) If f_n is absolutely continuous on $[0,1]$ for each $n = 1, 2, \cdots$. Prove or disprove that f is absolutely continuous on $[0,1]$. (10%)

4. Let λ, μ and ν be σ-finite measures on $(X, \mathcal{B})$. Assume that ν is absolutely continuous with respect to μ, and, μ is absolutely continuous with respect to λ.

   (a) If f is a nonnegative measurable function on X, prove that

   $$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu. \quad (10\%)$$

   (b) Prove that $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ almost everywhere with respect to λ. (5%)

5. (a) Prove or disprove that $L^p([0,1]) \supseteq L^q([0,1])$, where $1 \leq p < q < \infty$. (10%)

   (b) Prove or disprove that $l^q \supseteq l^p$, where $1 \leq p < q < \infty$, $l^p \equiv \{(x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ and $l^q \equiv \{(x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^q < \infty\}$. (10%)

6. (a) Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of closed subsets in $\mathbb{R}^n$. If D_1 is bounded and $D_n \supseteq D_{n+1}$ for all $n = 1, 2, \cdots$. Prove that $\bigcap_{n=1}^{\infty} D_n$ is nonempty. (5%)

   (b) Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of open dense subsets in $\mathbb{R}^n$. Prove that $\bigcap_{n=1}^{\infty} E_n$ is also dense in $\mathbb{R}^n$. (10%)

7. Let $F : l^2 \to \mathbb{R}$ be a bounded linear functional. Find the unique element $(a_1, a_2, a_3, \cdots)$ in l^2 such that

   $$F(x_1, x_2, \cdots) = \sum_{n=1}^{\infty} a_n x_n \quad \text{for any } (x_1, x_2, \cdots) \in l^2,$$

   and

   $$(\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}} = \sup\{|F(x_1, x_2, \cdots)| : \sum_{n=1}^{\infty} |x_n|^2 = 1\}. \quad (10\%)$$