國立中央大學數學系
博士班資格考試
〈分析〉試題
2007 年 2 月

There are 7 question sets of total 100 points.

Stage Setting: In the following problems, whenever not specified, the sets are assumed be Lebesgue measurable subsets of some Euclidean spaces $\mathbb{R}^n$ and integrations are Lebesgue integrals. We write $\mathscr{L}^n$ to be the Lebesgue measure of $\mathbb{R}^n$

1. 12 points Let $\delta \in (0,1)$. Suppose $\{A_k\}$ is a sequence of subsets of $[0,1]$ with its measure $\mathscr{L}^1(A_k) \geq \delta$. Show that there is a subset B of $[0,1]$ such that $\mathscr{L}^1(B) \geq \delta$ and every member $x \in B$ there are infinitely many k with $x \in A_k$. Furthermore, show that there is a subsequence $\{A_{k_j}\}$ of $\{A_k\}$ such that $\bigcap_{j=1}^{\infty} A_{k_j} \neq \emptyset$ with $k_j \nearrow \infty$ as $j \to \infty$.

2. 12 points Let $\Omega \subset \mathbb{R}^n$ with $0 < \mathscr{L}^n(\Omega) < \infty$ and let $f \in L^1(\Omega, \mathscr{L}^n)$ with f positive $\mathscr{L}^n$ almost everywhere on Ω. Is there a positive number λ so that both sets $\{x \in \Omega \mid f(x) \geq \lambda\}$ and $\{x \in \Omega \mid f(x) \leq \lambda\}$ whose corresponding Lebesgue measures are greater than or equal to one-half that of Ω? Justify your answer!

3. 12 points Let $I = [0,1)$, $\lambda \in I$, $\alpha, \beta \in \mathbb{R}$ and let

$$f(x) = \begin{cases} \alpha & \text{whenever } x \in [0,\lambda] \\ \beta & \text{whenever } x \in (\lambda, 1). \end{cases}$$

Suppose that f is extended to be defined on $\mathbb{R}$ satisfying $f(x+1) = f(x)$ and then define $f_k(x) = f(kx)$ for $k \in \mathbb{N}$. Show that, for any interval $(a,b) \subset \mathbb{R}$ and $g \in L^1((a,b), \mathscr{L}^1)$,

$$\int_{(a,b)} f_k(x) g(x)\, d\mathscr{L}^1 x \longrightarrow [\lambda\alpha + (1-\lambda)\beta] \int_{(a,b)} g(x)\, d\mathscr{L}^1 x \quad \text{as } k \to \infty.$$

4. 18 points Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{4xy - x^2 - y^2}{(x+y)^4} & \text{whenever } x > 0 \text{ and } y > 0 \\ 0 & \text{whenever } x \leq 0 \text{ or } y \leq 0. \end{cases}$$