Ph.D. Qualifying Examination: Analysis (2009.02)

Stage Setting: In the following problems, whenever not specified, the functions are assumed be real-valued.

1. (15%) Let $(X, \mathcal{B}, \mu)$ be a finite measure space and f be a nonnegative measurable function on X. Prove that there exists a Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions in $L^1(X, \mu)$ such that $f_n \nearrow f$ almost uniformly if and only if $\sup\{\int_X \varphi d\mu : \varphi \text{ is simple and } \varphi \leq f\} < \infty$.

2. (10%) Let μ be the Lebesgue measure on $[0, 1]$, ν be the counting measure on $[0, 1]$ and
$$\triangle = \{(x, x) : x \in [0, 1]\}.$$
Determine whether the characteristic function $\chi_\triangle$ is $\mu \times \nu$-integrable on $[0, 1] \times [0, 1]$. Justify your answer.

3. (15%) Let $\{\mu_n\}$ be an increasing sequence of measures defined on a measure space $(X, \mathcal{B})$, that is, $\mu_n(A) \leq \mu_{n+1}(A)$ for all $n \geq 1$ and $A \in \mathcal{B}$. Define $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ for each $A \in \mathcal{B}$.
   (a) Prove that μ is a measure on $(X, \mathcal{B})$, and μ_n is absolutely continuous with respect to μ for all n.
   (b) Let f_n be the Radon-Nikodym derivative of μ_n with respect to μ. If $A \in \mathcal{B}$ with $\mu(A) < \infty$, prove that $f_n \nearrow 1$ almost everywhere (with respect to μ) on A.

4. (15%) For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let
$$K(x) = \begin{cases} \frac{1}{k} \cdot \exp(\frac{-1}{1-|x|^2}), & |x| < 1; \\ 0, & |x| \geq 1, \end{cases}$$
where $k = \int_{|x|<1} \exp(-1/(1-|x|^2))dx$ and $|x| = (\sum_{j=1}^n |x_j|^2)^{1/2}$. Let $K_\varepsilon(x) = \varepsilon^{-n} K(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. Define
$$f_\varepsilon(x) = \int_{\mathbb{R}^n} f(t) K_\varepsilon(x - t)dt, \quad \text{for all } x \in \mathbb{R}^n.$$
   (a) If $f \in L^1(\mathbb{R}^n)$, prove that $\|f_\varepsilon - f\|_{L^1} \to 0$, as $\varepsilon \to 0$. 9 %
   (b) Prove that $C_0^\infty(\mathbb{R}^n)$, the set of all infinitely differentiable functions vanishing at infinity, is norm dense in $L^1(\mathbb{R}^n)$. 6 %

5. (15%) Suppose μ is a finite measure on X and $f \in L^\infty(\mu)$ with $\|f\|_\infty > 0$. Prove that
$$\lim_{n\to\infty} \frac{\int_X |f|^{n+1} d\mu}{\int_X |f|^n d\mu} = \|f\|_\infty.$$

6. (15%) Let $(C([0, 1]), \|\cdot\|_1)$ be the normed space of all continuous real-valued functions on $[0, 1]$ with the norm $\|f\|_1 = \int_{[0,1]} |f|$, and
$$A = \{f \in C([0, 1]) : f(0) = 0, f \text{ is differentiable on } (0, 1) \text{ and } |f'| \leq 1\}.$$
Prove that the integral equation
$$f(x) = \frac{1}{2} \int_0^x e^{-xy} f(y)dy, \quad x \in [0, 1],$$
has a unique solution $f \in \overline{A}$, where $\overline{A}$ is the closure of A in $(C([0, 1]), \|\cdot\|_1)$.

7. (15%) Let f be a continuous function on $[0, 1]$. Prove that
$$\lim_{n\to\infty} \frac{\int_0^1 x^n f(x)dx}{\int_0^1 x^n dx} = f(1).$$