2011年一月博士班資格考，《實分析》試題 (2011/1/28)

請回答下列問題，並給予詳細的證明或解釋。共計 6 題，滿分 100 分。

(1) (17分) Let $\{f_\alpha\}_{\alpha\in I}$ be a family of real-value functions defined on $[a,b]$. Prove that if there is a constant $M>0$ such that $|f_\alpha(x)|\le M$ for all $x\in[a,b]$ and all $\alpha\in I$, then for any countable set E in $[a,b]$, there exists a sequence $\{f_{\alpha_n}\}$ such that for any $x\in E$, the $\lim\limits_{n\to\infty} f_{\alpha_n}(x)$ exists.

(2) (17分) Let $\{f_n\}_{n\in\mathbb{N}}$ be a increasing sequence of measurable functions defined on $[0,1]$. Prove that if f_n converges to f in measure on $[0,1]$, then $f_n(x_0)\to f(x_0)$ as $n\to\infty$, where x_0 is any continuous point of f.

(3) (17分) Suppose that g is a measurable function defined on the measurable set E. Prove that if $f\cdot g\in L^1(E)$ for any $f\in L^1(E)$, then g is a bounded function on $E\setminus Z$, where Z is a set of measure zero in E.

(4) (17分) Let f be a nonnegative measurable function on $[0,1]$. Prove that $f\in L^1([0,1])$ if and only if

$$\sum_{n=0}^{\infty} 2^n\big|\{x\in[0,1]:\, f(x)\ge 2^n\}\big|<\infty.$$

(5) (16分) Let $f_n\in L^1([0,1])$, $n\in\mathbb{N}$, satisfies the following conditions:
(i) $|f_n(x)|\le F(x)$ for all $n\in\mathbb{N}$ and $x\in[0,1]$, where $F\in L^1([0,1])$;
(ii) $\lim\limits_{n\to\infty}\int_{[0,1]} f_n(x)g(x)dx=0$ for any $g\in C([0,1])$.
Prove that for any measurable set E in $[0,1]$, $\lim\limits_{n\to\infty}\int_E f_n(x)dx=0$.

(6) (16分) Let f be a bounded and uniformly continuous function on $\mathbb{R}$ and each $K_n\in L^1(\mathbb{R})$, $n\in\mathbb{N}$, satisfy the following conditions:
(i) $\int_{\mathbb{R}}|K_n(x)|dx\le M$ for all $n\in\mathbb{N}$;
(ii) $\int_{\mathbb{R}}K_n(x)dx=1$ for all $n\in\mathbb{N}$;
(iii) for any $\delta>0$, $\lim\limits_{n\to\infty}\int_{|x|\ge\delta}|K_n(x)|dx=0$.
Prove that $(K_n*f)(x)$ converges to $f(x)$ uniformly on $\mathbb{R}$.