中央大學數學系博士生資格考：分析 (Aug. 31, 2012)

1. (15%) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on $\mathbb{R}$. Prove that the set of points where this sequence converges is an $F_{\sigma\delta}$.

2. (15%) Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number? Give your reason.

3. (20%) A transformation $y = Tx$ of $\mathbb{R}^n$ into itself is called a *Lipschitz transformation* if there is a constant c such that $|Tx - Tx'| \leq c|x - x'|$. Show that every Lipschitz transformation maps measurable sets into measurable sets.

4. Let $C_0^\infty(\mathbb{R}^n)$ be the class of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.
   (a) (10%) Construct a nontrivial function in $C_0^\infty(\mathbb{R}^n)$ whose support is a ball or an interval.
   (b) (10%) By (a), prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

5. (15%) Let $f \in L^1(\mathbb{R}^n)$ and

$$f^*(x) = \sup \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cube Q with edges parallel to the coordinate axes and center x. Show that there is a constant c independent of f and α such that

$$|\{x \in \mathbb{R}^n : f^*(x) > \alpha\}| \leq \frac{c}{\alpha} \int_{\mathbb{R}^n} |f|, \qquad \alpha > 0.$$

Does there exist a constant C independent of f such that $\int_{\mathbb{R}^n} f^* \leq C \int_{\mathbb{R}^n} |f|$? Give your reason.

6. (15%) If $f \in L^1(\mathbb{R}^n)$. Prove that

$$\text{(6.1)} \qquad \lim_{r \to 0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)dy = f(x), \quad \text{almost every } x \in \mathbb{R}^n,$$

where $Q(x,r)$ denotes n-dimensional cube centered at x with edges parallel to the coordinate axes and side-length r. Let $x = (x_1, \cdots, x_n)$ and $Q(x,r)$ be replaced by $[x_1, x_1 + r] \times \cdots \times [x_n, x_n + r]$. Is the equality (6.1) still valid? Give your reason.