敬請公布

中央大學數學系博士班資格考
實分析試題（九十五年二月）

第 1 – 6 題，每題 15 分；第 7 題 10 分。總分 100 分。

1. Let $f : [0,1] \mapsto [0,1]$ be continuous. Show that the graph of f has zero measure.

2. Write $\mathcal{C}_b(\mathbb{R},\mathbb{R})$ to indicate the space of functions $f : \mathbb{R} \mapsto \mathbb{R}$, which is continuous and bounded. Let $B = \{f \in \mathcal{C}_b(\mathbb{R},\mathbb{R}) \mid f(x) > 0 \text{ for all } x \in \mathbb{R}\}$. Is B open in $\mathcal{C}_b(\mathbb{R},\mathbb{R})$?

3. Let
$$\sigma(x) = \begin{cases} x^3 + \frac{x}{|x|}, & \text{if} \quad -1 \le x \le 1 \text{ and } x \ne 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$
Find the Lebesgue decomposition and Radon-Nikodym derivative of $d\sigma$ with respect to dx, where dx is the Lebesgue measure on [-1, 1].

4. Let $f(x) \ge 0$ be in $L^1(\mathbb{R})$ with Lebesgue measure such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $\varepsilon > 0$, let $f_\varepsilon = \frac{1}{\varepsilon} f(\frac{x}{\varepsilon})$ and let $\phi \in C_0(\mathbb{R})$, the continuous function on $\mathbb{R}$ with compact support. Recall
$$(h * g)(x) = \int_{-\infty}^{\infty} h(x-t)g(t)\,dt \qquad \text{for} \quad h, g \in L^1(\mathbb{R}).$$
Prove that $\phi * f_\varepsilon \to \phi$ uniformly as $\varepsilon \to 0$.

5. For $f \in L^1(\mathbb{R})$ define the *Fourier transform* $\hat{f}$ of f by
$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt, \qquad x \in \mathbb{R}.$$
Show that if f and g belong to $L^1(\mathbb{R})$, then $\widehat{f * g}(x) = \hat{f}(x)\hat{g}(x)$.

6. Let $f \in L^1([a,b])$. Show that there is a set $A \subset (a,b)$ such that $\lambda(A^c \cap [a,b]) = 0$, where λ is the Lebesgue measure, and
$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} |f(t)-z|\,dt = |f(x)-z| = \lim_{h\to 0^+} \frac{1}{h}\int_{x-h}^{x} |f(t)-z|\,dt$$
for all $z \in \mathbb{C}$ and all $x \in A$.

7. Denote by $C(\mathbb{R})$ the linear space of all real valued continuous functions on $\mathbb{R}$. Show that the linear transformation $T : C(\mathbb{R}) \mapsto C(\mathbb{R})$ defined by
$$Tf(t) = \int_0^t f(x)\,dx$$
has no eigenvalue.