NCU Ph.D Qualification Exam. Real Analysis
Aug 2018

Note that you are being graded not only on the correctness but also on your presentation. There are **nine** problems on this test.

(1) **(8ps)** Let $B \subseteq \mathbb{R}^n$ be any set. Prove or disprove $\mathbb{R}^n \setminus \overline{B} = (\mathbb{R}^n \setminus B)^o$, where $\overline{B}$ denotes the closure of the set B and X^o denotes the interior of the set X.

(2) **(12ps)** Let $A \subset \mathbb{R}^n$ be a closed set. Given a continuous function $f : A \to [1, 2]$, define $g : \mathbb{R}^n \to \mathbb{R}$ by letting $g(x) = f(x)$ if $x \in A$ and for $x \in \mathbb{R}^n \setminus A$

$$g(x) \quad = \quad \frac{inf\, \{f(z)\, \|z - x\| \,|\, z \in A\}}{d(x, A)} \ ,$$

where in the above $d(x, A) = \inf\, \{\|x - a\| \,|\, a \in A\}$ denotes the distance from x to the closed set A. Prove or disprove that g is a continuous function on $\mathbb{R}^n$.

(3) **(7ps)** State Lusin's Theorem.

(4) **(7ps)** Define a measurable set in $\mathbb{R}^n$.

(5) **(6ps)** State Hölder's inequality.

(6) **(15ps)** Let $\{E_k\}$ be a sequence of sets with $\sum |E_k|_e < \infty$. Show that $\limsup E_k$ has measure zero. In the above $|\cdot|_e$ denotes the outer measure of a set.

(7) **(15ps)** Show that a set $E \subset \mathbb{R}^n$ is measurable if and only if for every set $A \subset \mathbb{R}^n$ we have $|A|_e \ = \ |A \cap E|_e \ + \ |A - E|_e$.

(8) **(15ps)**
Let $E \subset \mathbb{R}^n$ be any measurable set (not necessarily having finite measure). Fix $1 < p < \infty$. Suppose $f_k \to f$ *a.e.* on E, $f_k, f \in L^p(E)$ with $sup\{\|f_k\|_p \,|\, k \in \mathbb{N}\} \le M < \infty$, prove that for every $g \in L^q(E)$ where $1/p + 1/q = 1$ we have $\displaystyle\int_E f_k\, g \to \int_E f\, g$.

(9) **(15ps)**
Give an example of a function $f : (0, 1] \to \mathbb{R}$ which is **not** Lebesgue integrable on $[0, 1]$ but **improperly Riemann integrable** on $(0, 1]$. Of course, you need to prove that your example works.