NCU Ph.D Qualification Exam. Real Analysis

Aug 2011

State your assumptions or cite a known theorem if you are using it. Present your solutions as simple as possible yet rigorous enough. If not indicated, you can assume the measure/integration is Lebesgue measure/integration.

I) **Definitions and Concepts** (10 ps). Give the following definitions, there may be more than one definitions involved in some questions. In this case, give all relevant definitions and be as precise as possible.

(i) Let $\mathcal{S}$ be a fixed set. Give the definitions of $(i)\,\Sigma$, a σ-algebra of the subsets of $\mathcal{S}$ and (ii) an abstract measure μ on Σ.

(ii) Let f and ϕ be two functions that are finite and defined on $[a,b]$. Define the Riemann-Stieltjes integral of f with respect to ϕ:

$$\int_a^b f(x)d\phi(x) \ .$$

II) **Examples** (30 ps).

(i) Give an explicit example of a function $f : [0,1] \to \mathbb{R}$ such that f is continuous on $[0,1]$ but f is not of bounded variations (of course, you need to prove that your example works).

(ii) Give an explicit example of a function ϕ such that $(i)\,\phi \in C_o^\infty(\mathbb{R}^n)$, $(ii)\,\phi \ge 0$ on $\mathbb{R}^n$ and $(iii)\,\int_{\mathbb{R}^n} \phi\, dx = 1$ (of course, you need to prove that your example works).

III) **Theorems, computations and proof techniques.**

A) (15 ps) Let $\mathcal{C} = \{x \in (0,1)\,|\,\text{the decimal expansion of } x \text{ does not contain the digit } 8\}$. Compute the Lebesque measure of $\mathcal{C}$.

B) (15 ps) Let $f \in L^1_{loc}(\mathbb{R}^n)$ (this means that for any compact set $\Omega \subset \mathbb{R}^n$, $f \in L^1(\Omega)$). Suppose

$$\int_{\mathbb{R}^n} f\,\phi\, dx = 0 \ , \qquad \text{for all } \phi \in C_o^\infty(\mathbb{R}^n)$$

prove that $f = 0\, a.e.$. Note that the statement is trivial if f is continuous and therefore, if you establish the statement only for this case, you don't get any credit for it.

C) (15 ps) State and prove Vitali's covering lemma.

D) (15 ps) Given that $1 \le q \le p < \infty$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Prove that $\|\,\|f(x,y)\|_{L^q(dx)}\|_{L^p(dy)} \le \|\,\|f(x,y)\|_{L^p(dy)}\|_{L^q(dx)}$, that is

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|^q\,dx\right)^{\frac{p}{q}} dy\right)^{\frac{1}{p}} \le \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|^p\,dy\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} .$$