中央大學數學系博士生資格考：分析 (Aug. 30, 2013)

1. (a) (5%) Construct the Cantor set and show that it has measure zero.
   (b) (10%) Construct a singular function which is not absolutelely continuous.

2. (15%) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}^n$. Prove that $\{x \in \mathbb{R}^n : \lim_{k\to\infty} f(x) \text{ exists}\}$ is a measurable set.

3. Let

$$f(x) = \begin{cases} x\cos(1/x) & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^3\sin(1/x) & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0. \end{cases}$$

   (a) (7%) Determine whether f and g are of bounded variation on $[0, 1]$. Give your proof in each case.
   (b) (8%) Determine whether f and g are absolutely continuous functions on $[0, 1]$. Give your proof in each case.

4. (10%) Let $f \in L^1((0, 1))$, show that $x^k f(x) \in L^1((0, 1))$ for $k \in \mathbb{N}$ and $\int_0^1 x^k f(x)dx \to 0$.

5. (15%) Prove that $L^8(\mathbb{R}^n)$ is separable.

6. (15%) Let f be real-valued and mesurable on E, let $1 < p < \infty$ and $1/p + 1/p' = 1$. Prove that

$$\|f\|_p = \sup \int_E fg,$$

where the supremum is taken over all real-valued g such that $\|g\|_{p'} \leq 1$ and $\int_E fg$ exists.

7. (15%) Let $K \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} K(x)dx = 1$. If $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, show that

$$\|f * K_\varepsilon - f\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Here $K_\varepsilon(x) = \varepsilon^{-n}K(x/\varepsilon)$ and $(f * g)(x)$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$