Ph. D. Qualifying Examination: Real Analysis (2015.08.28)

1. Prove or disprove the following statements: (50%)

   (a) Let $f, f_1, f_2, \cdots$ be measurable functions on the measure space $(X, \mathcal{B}, \mu)$, and, $f_n \leq f_{n+1}$ for all $n \geq 1$. If $f_n \to f$ in measure, then $f_n \to f$ almost everywhere.

   (b) Let $f : [1, \infty) \to \mathbb{R}$ be continuous. If the improper Riemann integral $\int_1^\infty f(x)dx$ exists, then f is Lebesgue integrable on $[1, \infty)$.

   (c) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f' \in L^1[0, 1]$. Then

   $$\int_{[0,1]} f'(x)dx = f(1) - f(0).$$

   (d) Let $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Then f is Lebesgue integrable on $[0, 1] \times [0, 1]$.

   (e) Let $1 \leq p < q \leq \infty$ and $E \subseteq L^q([0, 1])$. Then the closure of E in $(L^q([0, 1]), \|\cdot\|_q)$ is contained in the closure of E in $(L^p([0, 1]), \|\cdot\|_p)$.

2. Find the value of the following limit:

   $$\lim_{n\to\infty} \int_0^\pi \left(1 - \frac{x}{n}\right)^n \sin x dx. \quad (10\%)$$

3. Let μ be the Borel measure defined by

   $$\mu(E) = \int_E \frac{dx}{(x^2 + 1)^2} \quad \text{for all Borel subsets } E \text{ of } \mathbb{R}.$$

   Prove that

   $$\int_{\mathbb{R}} f(x)d\mu(x) = \int_{\mathbb{R}} \frac{f(x)}{(x^2 + 1)^2} dx$$

   for all nonnegative Borel measurable functions f on $\mathbb{R}$. (10%)

4. Let ν be the Lebesgue measure on $[0, 1]$, and let μ be the counting measure on $[0, 1]$. Can there exist a μ-measurable function f such that

   $$\nu(E) = \int_E f d\mu \quad \text{for all Lebesgue measurable subsets } E \text{ of } [0, 1]?$$

   If such f exists, give your proof; otherwise, show that there does not exist such f. (10%)

5. Let Ω be a bounded domain in R^n, $n \geq 2$.

   (a) Can there exist a positive constant C_1 such that

   $$\Big(\int_\Omega f^4\Big)^3 \leq C_1 \Big(\int_\Omega f^6\Big)^2 \quad \text{for all } f \in L^6(\Omega)?$$

   (b) Can there exist a positive constant C_2 such that

   $$\Big(\int_{\mathbb{R}} g^4\Big)^3 \leq C_2 \Big(\int_{\mathbb{R}} g^6\Big)^2 \quad \text{for all } g \in L^4(\mathbb{R}) \cap L^6(\mathbb{R})?$$

   Prove or disprove your answer. (10%)

6. Let $\ell^2 \equiv \{\{a_n\}_{n=0}^\infty : \sum_{n=0}^\infty |a_n|^2 < \infty\}$ and $T \in (\ell^2)^*$ with $T(e_n) = 0$ for all $n \geq 1$, where e_n is the sequence with 1 at the nth place and 0 otherwise. Prove that there is some constant c such that $T(\{a_n\}_{n=0}^\infty) = ca_0$ for all $\{a_n\}_{n=0}^\infty \in \ell^2$. (10%)