Qualifying Examination
January 2013

Real Analysis

(Choose 5 to complete your answer.)

1. Prove that: Let μ^* be the outer measure on $R^d, d \geq 1$. Then the set $M(\mu^*)$ of all μ^*-measurable subsets of R^d is a σ-algebra.

2. True or False:

   (a) Let f and f_k, $k = 1, 2, ...$ be measurable and finite a.e. in E. If $f_k \to f$ a.e. on E, then $f_k \to f$ in measure on E.

   (b) If $f_k \to f$ in measure on E, then $f_k \to f$ a.e. on E.

   If ture, show the proof, if false, give a counterexample.

3. If $a < f(x) \leq b$ (a and b are finite) for $x \in E$, then show that
$$\int_E f = -\int_a^b \alpha d\omega(\alpha)$$
where ω is a distribution function of f on E.

4. Prove the following generalization of Hölder's inequality: if $\sum_{i=1}^k \frac{1}{p_i} = \frac{1}{r}$ and $p_i, r \geq 1$ then $||f_1 \cdots f_k||_r \leq ||f_1||_{p_1} \cdots ||f_k||_{p_k}$.

5. (a) Suppose that $f_k \to f$ a.e. and that $f_k, f \in L^p$, $1 < p < \infty$. If $||f_k||_p \leq M < +\infty$, show that $\int f_k g \to \int fg$ for all $g \in L^q$, $\frac{1}{p}+\frac{1}{q} = 1$.

   (b) A sequence $\{f_k\}$ in L^p is said to converge weakly to a function f in L^p if $\int f_k g \to \int fg$ for all $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f_k \to f$ in L^p norm, $1 \leq p \leq \infty$, then $\{f_k\}$ converges weakly to f in L^p. Given an example to state that the converse is not true.

6. Use Minkowski's integral inequality to prove : if $1 \leq p \leq \infty, f \in L^p(R^n)$, and $g \in L^1(R^n)$, then $f * g \in L^p(R^n)$ and $||f * g||_p \leq ||f||_p ||g||_1$.

7. Let $\phi(x), x \in R^n$, be a bounded measurable function such that $\phi(x) = 0$ for $x \geq 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$. ($\phi_\varepsilon$ is called an approximation to the identity.) If $f \in L(R^n)$, show that $\lim_{\varepsilon \to 0}(f * \phi_\varepsilon)(x) = f(x)$ in the Lebesgue set of f. [Note that $\int \phi_\varepsilon = 1$ when $\varepsilon > 0$, so that $(f * \phi_\varepsilon)(x) - f(x) = \int [f(x-y) - f(x)]\phi_\varepsilon(y)dy$.]