中央大學數學系博士生資格考：分析 (Jan. 24, 2014)

1. (15%) A set of $E$ is said to be Lebesgue measurable if for each set $A$ we have $|A|_e = |A \cap E|_e + |A \backslash E|_e$, where $|A|_e$ is denoted by Lebesgue outer measurable of $A$. Show that every set of Lebesgue outer measure zero is measurable.

2. (15%) Suppose $f$ is a measurable function on $\mathbb{R}$ and $B$ is a Borel subset of $\mathbb{R}$. Prove $f^{-1}(B)$ is a measurable set.

3. (15%) Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{for } 0 < x \le 1, \\ 0 & \text{for } x = 0. \end{cases}.$$

Is $f$ of bounded variation on $[0,1]$? Give your reason.

4. (15%) Use Fubini's theorem to prove that $\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$.

5. (10%) Let $f$ be absolutely continuous on $[0,2]$, prove or disprove that $f$ is of bounded variation on $[0,2]$.

6. (15%) Let $f \in L^3((0,1))$, show that $x^k f(x) \in L^1((0,1))$ for $k \in \mathbb{N}$ and $\int_0^1 x^k f(x) dx \to 0$.

7. (15%) Let $\phi$ be a bounded measurable function on $\mathbb{R}^n$ such that $\phi(x) = 0$ for $|x| \ge 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $f \in L^1(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0} (f * \phi_\varepsilon)(x) = f(x) \qquad \text{in the Lebesgue set of } f.$$

Here $(f * g)(x)$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$