DOCTORAL PROGRAM QUALIFYING EXAM : REAL ANALYSIS
2015/02/06

1. Let Z be a measure zero set in $\mathbb{R}^1$. Show that the set $\{x^2 : x \in Z\}$ has measure zero.

2. If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k \in Z} 2^{kp}\omega(2^k) < \infty$, where ω is the distribution function.

3. Given $f \in L^p(\mathbb{R}^1)$ for some $1 \leq p < \infty$. Given $t \in \mathbb{R}^1$, define $f_t(x) := f(x-t)$. Show that $\lim_{t \to 0} ||f_t - f||_p = 0$.

4. Suppose a function $f(x)$ on $[0,1]$ is defined as the following. If $x \in [0,1]$ is a rational number, i.e. $x = \frac{q}{p}$ for some integers p and q, then define $f(x) = \frac{1}{p}$, when x is not rational, define $f(x) = 0$. Prove or disprove f is measurable.

5. Given a set $E \subset \mathbb{R}^2$. Define $E^{++} = \{(x,y) : (x,y) \in E, x, y \geq 0\}$, $E^{+-} = \{(x,y) : (x,y) \in E, x \geq 0, y \leq 0\}$, and similarly for E^{-+} and E^{--}. Is the following true ? E is measurable if and only of all the four sets $E^{++}, E^{+-}, E^{-+}, E^{--}$ are measurable. Prove or disprove your result.

6. Denote f^* the Hardy-Littlewood maximal function of f. Given $f \in L^2(\mathbb{R}^n)$, show that $||f^*||_2 \leq c||f||_2$ for some constant c only depending on the dimension n.

7. Suppose μ and ν are two positive finite measures on the measurable space (X, Σ). Show that there exists a measurable function f on X such that for all $E \in \Sigma$ we have $\int_E (1-f) d\mu = \int_E f d\nu$.

8. Given a measurable set $E \subset [0,1]$ with $|E| = \frac{1}{2015}$. Show that E contains an arithmetic progression of length 3, i.e. there are three numbers $a, b, c \in E$ such that $b = a + r$ and $c = a + 2r$ for some $r \neq 0$. (Hint: use Lebesgue Differentiation Theorem)