NCU Ph.D Qualification Exam. Real Analysis

Feb 2018

State your assumptions or cite a known theorem if you are using it. Present your solutions as simple as possible yet rigorous enough.

I) **Definitions and Concepts** (10ps for each problem, total 30 ps). In the case that there may be more than one definitions involved in some questions, you should give all relevant definitions and be as precise as possible.

(i) Give the definition of a compact set in $\mathbb{R}^n$. Note that the **Heine-Borel** Theorem in $\mathbb{R}^n$ is **NOT** considered as the definition.

(ii) Let f and ϕ be two functions that are finite and defined on $[a, b]$. Define the Riemann-Stieltjes integral of f with respect to ϕ:

$$\int_a^b f(x) d\phi(x) \, .$$

(iii) Define a measurable set in $\mathbb{R}^n$.

II) **Examples** (10ps for each problem, total 30 ps).

(i) Give an explicit example of a function $f : [0, 1] \to \mathbb{R}$ such that f is continuous on $[0, 1]$ but f is not of bounded variations (of course, you need to prove that your example works).

(ii) Give an explicit example (include detail descriptions, not just the name of such a well known object) of an uncountable set whose (outer) measure is zero (of course, you need to prove that your example works).

(iii) Give an example of a decreasing sequence of nonempty closed sets in $\mathbb{R}$ whose intersection is empty.

III) **Theorems, computations and proof techniques.**

A) (10 ps) Prove that the distance between two non-empty, compact, disjoint sets in $\mathbb{R}^n$ is positive. (Recall: for $A, B \subset \mathbb{R}^n$, $d(A, B) = inf\{\|a - b\| \,|\, a \in A, b \in B\}$.)

B) (15 ps) Let $I = \{(x_1, x_2, ..., x_n) \,|\, a_i \leq x_i \leq b_i, i = 1, ..., n\}$ be a (closed) interval in $\mathbb{R}^n$. Denote by $v(I)$ the volume of I, that is $v(I) = (b_1 - a_1) \cdots (b_n - a_n)$. Show that $|I|_e = v(I)$ where $|I|_e$ denotes the outer measure of I (of course, you need to know the definition of an outer measure here).

C) (15 ps) Show that a set $E \subset \mathbb{R}^n$ is measurable if and only if for every set $A \subset \mathbb{R}^n$

$$|A|_e \;=\; |A \cap E|_e \;+\; |A - E|_e \, .$$

In the above $|\cdot|_e$ denotes the outer measure of a set.

End of questions