NCU PH.D. QUALIFICATION EXAMINATION
DIFFERENTIAL GEOMETRY

FEBRUARY 17 2005

Show your works in details. In each problem, you may assume the former part(s) and work on the latter part(s) directly.

1. (a) Let $f(x_1, \ldots, x_n)$ be C^∞ over $B_0(r)$ with $f(0) = 0$. Show that there are $g_i \in C^\infty(B_0(r))$, $i = 1, \ldots, n$, such that $f(x) = \sum_{i=1}^n x_i g_i(x)$.
   (b) Use (a) to show that an $\mathbb{R}$-linear map $F : C^\infty(TM) \to C^\infty(TM)$ is a tensor if and only if $F(fv) = fF(v)$ for all $f \in C^\infty(M)$, $v \in C^\infty(TM)$.
   (c) State and prove Cartan's formula for $d\omega$ where $\omega \in A^2(M)$.
   (d) Let ω be a 2-form. Define $E_p := \{\, v \in T_pM \mid \omega(v,u) = 0,\ \forall\, u \in T_pM \,\}$. Assume $d\omega = 0$ and E_p's are equi-dimensional for all $p \in M$. Show that the distribution $\{E_p\}_{p \in M}$ is integrable.

2. (a) Define the Lie derivative. For two C^∞ vector fields V, W with compact support on a manifold, show that $L_V W = [V, W]$.
   (b) For $\alpha \in A^p(M)$, show that $L_V \alpha = (\iota_V d + d\iota_V)\alpha$.

3. (a) Derive the the second variation formula for geodesics.
   (b) Prove the Bonnet-Myer Theorem: Let M be a complete Riemannian manifold of dimension n. If $\mathrm{Ric} \geq (n-1)k$ with $k > 0$ being a constant, then M is compact with $\mathrm{diam}\, M \leq \pi/\sqrt{k}$.

4. (a) State and prove Gauss' equation for submanifolds.
   (b) Let $\bar{M}$ be a compact oriented three dimensional manifold and M be an oriented minimal surface in $\bar{M}$ with unit normal vector field N. Show that $\mathrm{Ric}_{\bar{M}}(N,N) = (R_{\bar{M}} - |B|^2)/2 - K_M$ on M.
   (c) If $\bar{M}$ has positive scalar curvature ($R_{\bar{M}} > 0$), show that there is no stable oriented minimal surface M in $\bar{M}$ with $g(M) \geq 1$.

   (You may use the second variation formula of area with variation field $V = uN$, $u \in C^\infty(M)$: $A''(0) = \int_M |\nabla u|^2 - u^2(\mathrm{Ric}_{\bar{M}}(N,N) + |B|^2)$).

5. (a) Use the Stokes' theorem to show that there is a well-defined map $T_M : H^p_{dR}(M) \to H^p(M; \mathbb{R})$ which commutes with pull backs.
   (b) For M compact, show that a smooth representative $\alpha \in A^p(M)$ of a de Rham cohomology class has minimal length if and only if $\triangle\alpha = 0$.
   (c) State the Hodge decomposition theorem and use it to show that each cohomology class contains an unique minimal representative.

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