敬請公布

NCU PH.D. QUALIFICATION EXAMINATION
DIFFERENTIAL GEOMETRY

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Show your work in details. In each problem, you may assume the former part(s) and work on the latter part(s) directly.

(15%) 1. Let $S \subset \mathbb{R}^3$ be a smooth surface. Up to a translation and a rotation of $\mathbb{R}^3$, a neighborhood of $p \in S$ is the graph of a function $z = h(x, y)$ with $h(0,0) = p$, $h_x(0,0) = 0 = h_y(0,0)$.

(a) Use this representation to compute the second fundamental form II_p of S at p.

(b) Find the Gaussian curvature K_p and mean curvature H_p of S at p.

(c) Show that if S is a minimal surface then $K_p \leq 0$, $\forall p \in S$.

(25%) 2. (a) Consider the torus of revolution generated by rotating the circle

$$(x-a)^2 + z^2 = r^2, \ y = 0$$

about the z-axis ($a > r > 0$). Show that the circle generated by the point $(a + r, 0) \in \mathbb{R}^2_{x,z}$ is a geodesic on the torus of revolution.

(b) State Gauss-Bonnet Theorem for an orientable compact surface.

(c) Compute the Euler-Poincaré characteristic $\chi(T)$ of a torus T via a triangulation of T.

(d) Is it possible to embed T into $\mathbb{R}^3$ so that its Gaussian curvature $K \equiv 0$ in the induced metric? Justify your answer.

(e) Give an example of a flat torus and verify that its curvatrure $K \equiv 0$.

(10%) 3. (a) State and prove Cartan's formula for $d\omega$ where $\omega \in \Omega^2(M)$ is a C^∞ 2-form on a differentiable manifold M.

(b) Let ω be a 2-form. Define $N_p := \{v \in T_pM \mid \omega(v, u) = 0, \forall u \in T_pM\}$. Assume that $d\omega = 0$ and $N_p's$ are equi-dimensional for all $p \in M$. Show that the distribution $\{N_p\}_{p\in M}$ is integrable.

(10%) 4. Let V be a C^∞ vector field with compact support on a manifold M. Show that $L_V\alpha = (\iota_V \circ d + d \circ \iota)\alpha$, $\forall \alpha \in \Omega^p(M)$.

(6%) 5. Let α be a 1-form on the unit sphere $S^2 \subset \mathbb{R}^3$. Show that if α is $SO(3)$-invariant then $\alpha \equiv 0$.

(14%) 6. Let M be a connected n-dimensional compact Riemannian manifold with the fundamental group $\pi_1(M) = 0$.

(a) Show that if $\alpha, \beta \in \Omega^n(M)$ satisfy $\int_M \alpha = \int_M \beta$ then α and β differ by an exact 1-form.

(b) Show that the only harmonic 1-form on M is 0.

(20%) 7. (a) Derive the second variation formula for geodesics.

(b) Prove the Bonnet-Myer Theorem: Let M be a complete Riemannian manifold of dimension n. If $\text{Ric} \geq \frac{n-1}{r^2}$ with $r > 0$ being a constant, then M is a compact with diam $M \leq \pi r$.

(c) Show that if M is a complete Riemannian manifold with $\text{Ric} \geq \delta > 0$, then the universal cover of M is compact and in particular, the fundamental group $\pi_1(M)$ is finite.