國立中央大學數學系九十三年度第二學期博士班資格考試微分方程試題

1. (20%) Consider the system $\dot{x} = f(x)$ with $f \in C^1(E)$, where E is an open subset of $\mathbb{R}^n$. Let Γ be a trajectory of the system in a compact subset of $\mathbb{R}^n$. Show that the ω-limit set of Γ is a non-empty, closed, connected and compact subset of E.

2. (20%)

   (a). State the Liouville's theorem.

   (b). Let $E \subseteq \mathbb{R}^n$, $f \in C^1(E)$ and $u(t,y) \in C^1(G)$ be a solution of the initial value problem:

$$\dot{x} = f(x) \quad \text{and} \quad x(0) = y.$$

   where $G = [-a, a] \times N_\delta(x_0)$, $a > 0, x_0 \in E$ and $\delta > 0$. Show that for all $t \in [0, a]$

$$\det \frac{\partial u}{\partial y}(t, x_0) = \exp \int_0^t \nabla \cdot f(u(s, x_0))ds.$$

3. (20%)

   (a). State the Poincaré-Bendixson theorem in $\mathbb{R}^2$.

   (b). Show that there is a periodic orbit of the following system:

$$\dot{x} = -y + x(r^4 - 3r^2 + 1) \quad \text{and} \quad \dot{y} = x + y(r^4 - 3r^2 + 1).$$

   in the annular region $A = \{x \in \mathbb{R}^2 | 1 < |x| < 3\}$. Here $r^2 = x^2 + y^2$.

4. (20%)

   (a). State the Floquet's theorem.

   (b). Let $\gamma(t) = (\cos t, \sin t, 0)^T$ be a periodic orbit of the following system:

$$\dot{x} = x - y - x^3 - xy^2, \quad \dot{y} = x + y - x^2y - y^3 \quad \text{and} \quad \dot{z} = \lambda z.$$

   Show that $\Phi(t)$ defined by

$$\Phi(t) = \begin{pmatrix} e^{-2t}\cos t & -\sin t & 0 \\ e^{-2t}\sin t & \cos t & 0 \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$$

   is a fundamental matrix of the linearization of the above system about the periodic orbit $\gamma(t)$.

   (c). Find the characteristic exponents of $\gamma(t)$.

5. (20%)

   (a). State the theorem of Dulac's Criteria.

   (b). Use the Dulac function $B(x,y) = be^{-2\beta x}$ to show that the system

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -ax - by + \alpha x^2 + \beta y^2$$

   has no limit cycle in $\mathbb{R}^2$.