Differential Equations （微分方程）

1. (20%) Let D be an open set in $R \times R^{n+1}$ with an element of D written as (t, x), and $f : D \to R^n$ be a continuous function. Consider the following differential equation

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases} \tag{1}$$

Prove the following statements respectively :

(a) For any $(t_0, x_0) \in D$ there is at least one solution of (1) passing through (t_0, x_0).

(b) If, in addition, $f(t, x)$ is locally lipschotzian with respect to x in D, then for any (t_0, x_0) in D, there exists a unique solution $x(t, t_0, x_0)$ of (1) passing through (t_0, x_0).

2. (20%) Consider the linear system

$$x'(t) = A(t)x(t), \tag{2}$$

where $A(t) = (a_{ij}(t))_{n\times n}$ and the a_{ij} for $i, j = 1, \cdots n$ are continuous real valued functions on the interval $(-\infty, +\infty)$.

(a) Suppose there is a constant K such that a fundamental matrix X of Eq. (2) satisfies $||X(t)|| \le K, t \ge \beta$ and $\liminf_{t\to\infty} \int_\beta^t \text{trace}(A(s))\, ds > -\infty$. Prove that X^{-1} is bounded on $[\beta, \infty)$ and no nontrivial solution of Eq. (2) approaches zero as $t \to \infty$.

(b) Let $B(t)$ be a continuous real $n \times n$ matrix for $t \ge \beta$ with $\int_\beta^\infty ||A(t) - B(t)||dt < \infty$. Prove that every solution of

$$y'(t) = B(t)y(t) \tag{3}$$

is bounded on $[\beta, \infty)$. For any solution x of Eq. (2), prove there is a unique solution y of Eq. (3) such that $x(t) - y(t) \to 0$ as $t \to \infty$.

3. (20%) Consider the van der Pol equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0. \tag{4}$$

Let Γ be the unique asymptotically stable limit cycle for every $\epsilon > 0$. Let $D = \{(x, y)|x^2 + y^2 < 3\}$. Prove $\Gamma \not\subset D$. (State the theorem used.)

4. Consider the following Predator-Prey system

$$\begin{cases} \dfrac{dx}{dt} = x(\gamma(1 - \dfrac{x}{K}) - \dfrac{my}{a + x}) \\ \dfrac{dy}{dt} = (\dfrac{mx}{a + x} - d)y, \qquad\qquad \gamma, K, m, a, d > 0 \\ x(0) > 0, y(0) > 0. \end{cases} \tag{3}$$

For various possible cases, do the following :

(a) (15%) Do the stability analysis for each equilibrium with nonnegative components.

(b) (5%) Find the stable manifold of each saddle point.

5. Let $v(t, a)$ be the solution of

$$v''(t) + t \sin(v(t)) = 0,$$
$$v'(0) = 0, v(0) = a.$$

Prove that $v(t, a)$ is oscillatory over $[0, \infty)$