Qualify Exam for Ordinrary Differential Equations

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1.(10%)

Let E be a normed vector space, $W \subset \mathbf{R} \times E$ an open set, and $f, g : W \to E$ continuous. Suppose that for all $(t, x) \in W$, $|f(t, x) - g(t, x)| < \varepsilon$. Let K be a Lipschitz constant in x for $f(t, x)$. If $x(t), y(t)$ are solutions to $x' = f(t, x)$ and $y' = g(t, y)$, respectively, on some interval J, $t_0 \in J$ and $x(t_0) = y(t_0)$. Show

$$|x(t) - y(t)| \leq \frac{\varepsilon}{K}\{e^{K|t-t_0|} - 1\}.$$

2.(20%)

Consider the system

$$x' = -y + x(1 - x^2 - y^2),$$
$$y' = x + y(1 - x^2 - y^2).$$

(1) (10%) Find the Poincaré map for the periodic solution $\Gamma(t) := (\cos t, \sin t)^T$.

(2) (10%) Use the Poincaré map to determine the stability of $\Gamma(t)$.

3. (15%)

Consider the following system:

$$x' = x - y - x^3 \quad y' = x + y - y^3.$$

Show that there is at least one stable limit cycle in the region $A := \{(x, y) | \ 1 \leq |(x, y)| \leq \sqrt{2}\ \}$.

4. (15%)

(1) (5%) State the Floquet's Theorem.

(2) (10%) Prove the statement (which is given by you) in part (1).

5. (30%)

Consider the system

$$
\begin{aligned}
x' &= -y + x(1 - x^2 - y^2), \\
y' &= x + y(1 - x^2 - y^2), \\
z' &= \lambda z,
\end{aligned}
$$

where λ is a constant. Let $\gamma(t) = (\cos t, \sin t, 0)^T$ be a periodic orbit of the system.

(1) (5%) Find the linearization of the system about $\gamma(t)$.

(2) (10%) Find the fundamental matrix $\Phi(t)$ for the linearized system which satisfies $\Phi(0) = I$.

(3) (15%) Compute the characteristic multiplier of $\gamma(t)$ and dimension of stable, unstable and center manifolds of $\gamma(t)$.

6. (10%)

Find solution of the following initial value problem:

$$
x'(t) = \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t), \quad x(0) = x_0,
$$

for some constant vector x_0.