Differential Equations

Ph.D. Qualify Exam for the Dept. of Math, Nat'l Central Univ.

Problem 1. Complete the following.

1. (10%) Consider an one-parameter family of linear equations

$$x'(t) = A(\alpha, t)x(t)\,,$$

where A is a continuous function of t and α. Show that the flow $\phi(\alpha, t, x)$ depends continuously on x and α.

2. (10%) Consider the equations of variation

$$x'(t) = f\big(t, x(t)\big)\,,$$
$$y'(t) = (Df)\big(t, x(t)\big)y(t)\,.$$

with a C^1 velocity field f. Show that the equations above have a unique solution which depends continuously on $x(0)$ and $y(0)$.

Problem 2. (10%) Let $v(t)$ be a continuously differentiable function satisfying the differential inequality

$$v'(t) \le c(t) + \int_0^t u(s)v(s)ds\,, \qquad v(0) = v_0\,,$$

where $c(t)$, $u(t)$ are both non-negative continuous functions. Let $w(t)$ satisfy the differential equation

$$w'(t) = c(t) + \int_0^t u(s)w(s)ds\,, \qquad w(0) = v_0\,.$$

Prove that $w(t) \ge v(t)$ for $t \ge 0$. Is it necessary that both $c(t)$ and $u(t)$ be non-negative for the above statement to hold?

Problem 3. For the ODE $x'(t) = f(x)$, we consider the corresponding first order partial differential operator $\mathcal{A} = f(x) \cdot \nabla x$ acting on vector-valued functions $u(x)$ (componentwise).

1. (5%) Suppose that $f(x) = \mathrm{M}x$ with M a $d \times d$ matrix. Verify that $e^{tA}x = e^{tB}x$. Is it true in general that $e^{tA}u(x) = e^{t\mathrm{M}}u(x)$?

2. (10%) The first integrals of an ODE are real-valued functions $U(t,x)$ such that $U(t,x(t))$ do not change in time if $x(t)$ is a solution to that ODE. In other words, the first integrals are certain invariants of the system. Show that any solution $U(t,x)$ to the PDE

$$\frac{\partial U}{\partial t} + f(t,x) \cdot \nabla_x U = 0$$

is a first integral of the ODE.

3. (10%) Use the method of characteristics to find the general form of the first integrals.

Problem 4. (15%) Find the form of the regions of stability in the (ϵ, ω)-plane for the system described by the equation

$$x'' = -f(t)x\,,$$

where f is a periodic function with period 2π, and

$$f(t) = \begin{cases} \omega + \epsilon & \text{for } 0 \le t < \pi\,, \\ \omega - \epsilon & \text{for } \pi \le t < 2\pi\,, \end{cases} \qquad \epsilon \ll 1\,.$$

Problem 5. (15%) Consider the nonlinear oscillator

$$x'' + cx' + ax + bx^3 = 0\,,$$

where a, b and c are positive constants, and $y = x'$. Show that $(x,y) = (0,0)$ is Liapounov stable using Liapounov function of the form $V(x,y) = \alpha x^2 + \beta x^4 + \gamma y^2$ for some positve constants α, β and γ.

Problem 6. (15%) Let $c > 0$. Solve the boundary value problem

$$u(x) - u''(x) = \delta(x - c) \qquad \text{for} \quad x > 0\,, \tag{0.1a}$$

$$u(0) = 0\,, \tag{0.1b}$$

$$u(\infty) \equiv \lim_{x\to\infty} u(x) = 0\,, \tag{0.1c}$$

where δ is the Dirac delta function (so $\delta(x-c)$ is the Dirac delta function at the point c).