博士班資格考科目	考試時間	節次	頁數
Differential Equations	8 月 24 日	9:00 am — 12:00 pm	共 1 頁

1. (A) (12 points) Consider the following initial value problem

$$(1)\begin{cases} y'' + (y^2+1)y' + \sin y = 0, \\ y(0) = 1, \quad y'(0) = 1. \end{cases}$$

Please show in detail that the solution of (1) exists uniquely and locally.

(B) (13 points) Consider the following initial value problem

$$(2)\begin{cases} x' = x(1-\frac{x}{2}) - xy, \\ y' = y(x-2), \\ x(0) > 0, \quad y(0) > 0. \end{cases}$$

Show that the solutions $x(t)$, $y(t)$ exist for all $t > 0$, and the solutions are positive and bounded for all $t > 0$.

2. (15 points) Solve the initial value problem

$$\begin{cases} x' = Ax, \\ x(0) = x_0, \end{cases}$$

where

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

Determine the stable and unstable subspaces and sketch the phase portrait.

3. (20 points) Consider the following initial value problem

$$\begin{cases} x' = x(1-\frac{x}{2}-y), \\ y' = (2x-1-2y)y, \\ x(0) > 0, \quad y(0) > 0. \end{cases}$$

(A) Find all equilibria with nonnegative components.

(B) Do stability analysis for each equilibrium.

(C) Find the stable manifold of each saddle point.

(D) Predict the global asymptotic ($t \to \infty$) behavior.

4. (20 points) Consider the following boundary value problem

$$(3)\begin{cases} u'' = k(x)\sin^2 u, & \text{in} \quad (0,1) \\ u(0) = A, \\ u(1) = B, \quad A, B \in \mathbb{R} \end{cases}$$

where k is a continuous function of x satisfying $|k(x)| \le 1$ for all $x \in [0,1]$. Find the Green function of (3), and use contraction mapping theorem to show the existence and uniqueness of solution for (3).

5. (20 points) Find the first three successive approximations $u^{(1)}(t,a)$, $u^{(2)}(t,a)$ and $u^{(3)}(t,a)$ for the stable manifold S of

$$\begin{cases} \dot{x_1} = -x_1 - x_2^2 \\ \dot{x_2} = x_2 + 2x_1^2, \end{cases}$$

near the origin, and use $u^{(3)}(t,a)$ to approximate S.