Ordinary Differential Equations 2012/08/31

1.(a)(15%) Consider the following initial value problem (IVP)

$$\begin{cases} \dfrac{dx}{dt} = F(t, x(t)), \quad f : D \subseteq R \times R^n \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where D is an open set of $R \times R^n$ containing (t_0, x_0). Prove the following statements respectively :

(a.1) For any $(t_0, x_0) \in D$ there is at least one solution of (1) passing through (t_0, x_0).

(a.2) If, in addition, $F(t, x)$ is locally lipschotzian with respect to x in D, then for any (t_0, x_0) in D, there exists a unique solution $x(t, t_0, x_0)$ of (1) passing through (t_0, x_0).

(b)(15%) Let $f : E \to E$ be continuous; suppose $f(x) \leq M$ $\forall x \in E$. For each $n = 1.2. \cdots$, let $x_n : [0, 1] \to E$ be a solution to $x' = f(x)$. If $x_n(0)$ converges, show that a subsequence of $\{x_n\}$ converges uniformly to a solution.

2. Consider the following Predator-Prey system

$$\begin{cases} \dfrac{dx}{dt} = x(\gamma(1 - \dfrac{x}{K}) - \dfrac{my}{a + x}) \\ \dfrac{dy}{dt} = (\dfrac{mx}{a + x} - d)y, \qquad \gamma, K, m, a, d > 0 \\ x(0) > 0, y(0) > 0. \end{cases} \tag{3}$$

For various possible cases, do the following :

(a) (15%) Do the stability analysis for each equilibrium with nonnegative components.

(b) (5%) Find the stable manifold of each saddle point.

3. (15%) Show that the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + (1 - 2x^2 - 3y^2)y \end{cases} \tag{4}$$

has at least one periodic solution.

4. (20%) Let A be a constant $n \times n$ real matrix. Prove the following statement respectively.
   (a) If every solution of $x' = Ax$ tends to 0 as $t \to \infty$, then every eigenvalue of A has negative real part.
   (b) Suppose that every eigenvalue of A has real part less than $-a < 0$. Then there exists a constant $K > 0$ such that if $x(t)$ is a solution to $x' = Ax$, then

$$|x(t)| \leq K^{-at}|x(0)|$$

for all $t \geq 0$. Find such a K and a if $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$.

5. (15%) Find the general solution of $u'(t) = Au(t) + g(t)$, where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{pmatrix}, \quad g(t) = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}.$$