QUALIFY EXAM FOR ODE

Department of Mathematics, National Central University

Ph D. program, Aug. 30, 2011

(1) (20%) Solve the following Sturm-Liouville problems, i.e., you need to compute the eigenvalues and eigenfunctions of each problem:

(a) Dirichlet boundary: $u'' + \lambda u = 0, \qquad u(0) = u(\pi) = 0$

(b) Neumann boundary: $u'' + \lambda u = 0, \qquad u'(0) = u'(\pi) = 0$

(c) Could you give an intuitive reason how the boundary effects the eigenfunctions?

(2) (15%) Prove the following general fact: if $C \geq 0$ and $u, v : [0, \beta] \to [0, \infty)$ are continuous and

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

for all $t \in [0, \beta]$, then

$$u(t) \leq Ce^{V(t)}, \qquad V(t) = \int_0^t v(s)ds$$

(3) (15%) Show that the van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x - \mu(x^2 - 1)y\,.$$

Investigate the stability properties of the critical point $(0, 0)$ for the cases $\mu > 0$ and $\mu < 0$.

(4) (15%) Let A be an $n \times n$ constant matrix. Show that the Picard method (Picard iteration) for solving $x' = Ax, x(0) = u$ gives the solution $e^{tA}u$. (You need to start from $x_0 = u$, then define the Picard iteration $x_{n+1} =$? and show $\lim_{n\to\infty} x_n = e^{tA}u$. You need to give a rigorous proof of the convergence.)

(5) (20%) Consider the nonlinear system

$$\frac{dx}{dt} = -2x + 2x^2, \qquad \frac{dy}{dt} = -3x + y + 3x^2$$

(a) Find out all the equilibrium points of this system.

(b) Classify the nature of the equilibrium point by linear analysis.

(c) Figure out the phase plane of this system.

(6) (15%) Consider a nonlinear autonomous system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y)\,, \tag{$*$}$$

in which the functions $F(x, y)$ and $G(x, y)$ are continuous and have continuous first partial derivatives through the phase plane. Show that if $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$ is always positive or always negative in a certain region of the phase plane, then the system $(*)$ cannot have a closed path in that region.
(Hint: Assuming we have a closed path $C = [(x(t), y(t)]$ and Ω is the region enclosed by C, then apply the Green's theorem to obtain a contradiction.)