## Ordinary Differential Equations

Please choose 5 problems to complete your answers in the following test.

1. Let $X(t, x_0)$ be the solution of the linear system $\begin{cases} X' = AX \\ X(0) = x_0, \end{cases}$ where $A \in R^{n \times n}$. Find $X(t, x_0)$ of the following problems respectively.

(1)

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

(2)

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix}$$

2. Consider the following initial value problem

$$\begin{cases} \frac{dx}{dt} = g(t, x), \\ x(t_0) = x_0, \end{cases} \qquad (IVP)$$

where $g : E \subset R \times R^n \to R^n$, and $E$ is an open set of $R \times R^n$ containing $(t_0, x_0)$. State and prove the local existence theorem of $(IVP)$. Furthermore, show the uniqueness result under suitable condition on $g$.

3. Consider the positive solution $(x(t), y(t))$ of the following equation

$$\begin{cases} x'(t) = [A - By(t)]x(t) \text{ on } [0, \infty), \\ y'(t) = [Cx(t) - D]y(t) \text{ on } [0, \infty), \\ x(0) = x_0 > 0, \; y(0) = y_0 > 0, \end{cases} \qquad (II)$$

where A, B, C and D are positive constants. How about $\lim_{t \to \infty} (x(t), y(t))$ with respect the constants $A, B, C, D$ ? Show your answer.

**4.** Consider the following initial value problem

$$\begin{cases} my''(t) + fy'(t) + ry(t) = 0 \text{ on } [0, \infty), \\ y(0) = y_0 > 0, \ y'(0) = y_1 > 0, \end{cases} \tag{$*$}$$

where m(mass), f(friction), and r(restoring force) are positive constants. Prove that every solution $y(t)$ of $(*)$ satisfies $\lim_{t\to\infty} y(t) = 0 = \lim_{t\to\infty} y'(t)$.

**5.** Discuss the solutions structure of the $(R - L - C)$ Model

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

with respect to the positive parameters $R, L, C$.

**6.** Consider the following predator-prey system

$$\begin{cases} x' = \gamma x(1 - \frac{x}{K}) - \alpha xy & , \\ y' = y(\beta x - d), & \gamma, K, \alpha, \beta, d > 0 \\ x(0) > 0, y(0) > 0. \end{cases}$$

Show that the solutions are positive and bounded for all $t > 0$. Discuss the asymptotical behaviors of $(x(t), y(t))$ with respect to the constants $\gamma, K, \alpha, \beta, d$.

**7.** Consider the population model

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{k}) - \frac{mx}{a + x}y, \\ \frac{dy}{dt} = sy(1 - \frac{y}{hx}), \\ x(0) > 0, y(0) > 0. \end{cases}$$

Show that the solutions $(x(t), y(t))$ are positive and bounded. Discuss the asymptotical behaviors of $(x(t), y(t))$ with respect to the positive constants $r, k, m, a, s, h$.