Ordinary Differential Equations

Please choose four problems to finish your answers respectively (from the following problems 1-7).

1. State and prove the local existence of solutions for the following (IVP)

$$\begin{cases} Y' = F(t, Y) \\ Y(t_0) = Y_0, \end{cases}$$

where F is a continuous function from the open set $A \subseteq R \times R^n$ to R^n and $(t_0, Y_0) \in A$.

2. State and prove the local uniqueness of solutions for the following (IVP)

$$\begin{cases} Y' = F(t, Y) \\ Y(t_0) = Y_0, \end{cases}$$

where F is Lipschitz continuous from the open set $A \subseteq R \times R^n$ to R^n and $(t_0, Y_0) \in A$.

3. Let $Y(t, Y_0)$ be the solution of the following initial value problem

$$\begin{cases} Y' = AY \\ Y(0) = Y_0, \end{cases}$$

where $A \in R^{n \times n}$ and $Y_0 \in R^n$.

(i) Show that: if all of the eigenvalues of A are distinct and have negative real parts, then for all $Y_0 \in R^n$ we have

$$\lim_{t \to \infty} Y(t, Y_0) = O.$$

(ii) Find $Y(t, Y_0)$ if

$$A = \begin{pmatrix} 4 & 1 \\ -8 & -5 \end{pmatrix}, \ Y_0 = \begin{pmatrix} 1 \\ -8 \end{pmatrix}.$$

4. Consider the following initial value problem

$$\begin{cases} X' = X(1 - Y - X) \\ Y' = Y(X - \frac{1}{2}), \\ X(0) = X_0 > 0, \ Y(0) = Y_0 > 0. \end{cases}$$

Discuss the asymptotic behaviors of all positive solutions.

5. Consider the following predator-prey system

$$\begin{cases} x' = \gamma x(1 - \frac{x}{K}) - \alpha xy & , \\ y' = y(\beta x - d), & \gamma, K, \alpha, \beta, d > 0, \\ x(0) = x_0 > 0, y(0) = y_0 > 0. \end{cases}$$

Discuss the stability properties of all equilibrium points for above equation.

6. Let $y(t)$ be the solution of the following initial value problem

$$\begin{cases} m_0 y''(t) + f_0 y'(t) + r_0 y(t) = 0 \ \text{ on } \ [0, \infty), \\ y(0) = y_0 > 0, \ y'(0) = y_1 > 0, \end{cases} \quad (*)$$

where m_0(mass), f_0(friction), and r_0(restoring force) are positive constants. Prove: $\lim_{t \to \infty} (y(t), y'(t)) = (0, 0)$.

7. Consider the following linear periodic system

$$Y' = A(t)Y, \quad (1)$$

where $A(t) = [a_{ij}(t)]_{n \times n} \in R^{n \times n}$ is continuous on R and $A(t) = A(t + T)$ is a periodic function with period T.

(i) State the definition of the Floquet multipliers (characteristic multipliers) and the Floquet Theorem of (1).

(ii) Explain that the Floquet multipliers are uniquely determined by the system (1).