1. (14%) Let $u : [0, b] \to \mathbb{R}^+$ is continuous that satisfies

$$u(t) \leqslant M + \int_0^t k(s)g(u(s))ds \text{ for } t \in [0, b]\,,$$

where $M \geqslant 0$, $k : [0, b] \to \mathbb{R}^+$ is continuous, g is continuous and monotone increasing. Show that

$$u(t) \leqslant \Phi^{-1}(\Phi(M) + \int_0^b k(s)ds) \text{ for } t \in [0, b]\,,$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is given by $\Phi(u) = \displaystyle\int_{u_0}^{u} \frac{ds}{g(s)}$ for some constant $u_0, u \in \mathbb{R}$.

2. (14%) Consider the following initial value problem

$$\begin{aligned} \dot{x} &= f(x,t) + \int_0^t g(x(s), s)ds \\ x(0) &= x_0\,, \end{aligned}$$

where f and g are Lipschitz continuous with respect to x in some neighborhood containing $(x_0, 0)$. Show that there exists $b > 0$ such that the solution exists uniquely in $(-b, b)$.

3. (10%) Solve the linear system

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \\ \dot{z} &= y \end{aligned}$$

with the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 = (x_0, y_0, z_0)^T$.

4. (a) (7%) Show that $\Phi(t) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}$ is a fundamental matrix solution of $\boldsymbol{x} = A(t)\boldsymbol{x}$ with $A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}$

   (b) (7%) Find the inverse of $\Phi(t)$ and solve the nonhomogeneous linear system $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} + b(t)$ with $A(t)$ given above and $b(t) = (1, e^{-2t})^T$.

5. (10%) Show that the system

$$\begin{aligned} \dot{x} &= \frac{y}{1+x^2} \\ \dot{y} &= \frac{-x + y(1 + x^2 + x^4)}{1+x^2} \end{aligned}$$

has no limit cycle on $\mathbb{R}^2$.

6. (12%) Show that there is a periodic orbit of the following system:

$$\begin{aligned} \dot{x} &= -y + x(r^4 - 3r^2 + 1) \\ \dot{y} &= x + y(r^4 - 3r^2 + 1) \\ r^2 &= x^2 + y^2 \end{aligned}$$

in the annular region $A = \{x \in \mathbb{R}^2 : 1 < |x| < 3\}$.

7. (12%) Do the stability analysis of the Brusselator equations:

$$\begin{aligned} \frac{dx}{dt} &= a - (b+1)x + x^2 y \,,\ a, b > 0 \\ \frac{dy}{dt} &= bx - x^2 y \end{aligned}$$

8. (a) (7%) Show that the system $\dot{x} = -x - xy^2$, $\dot{y} = -y - x^2 y$ is globally asymptotically stable, by guessing a suitable Liapunov function.

   (b) (7%) Consider the system

$$\begin{aligned} \dot{x} &= x^3 + yx^2 \\ \dot{y} &= -y + x^3 \,. \end{aligned}$$

   Show that $(0, 0)$ is an unstable equilibrium by using Lyapunov function

$$V(x, y) = \frac{x^2}{2} - \frac{y^2}{2} \,.$$