國立中央大學數學系九十三學年度博士班資格考試題卷

科目：常微分方程

1.(15%) Let r, k, and f be real and continuous functions which satisfy $r(t) \geq 0$, $k(t) \geq 0$, and

$$r(t) \leq f(t) + \int_a^t k(s)r(s)ds, \ a \leq t \leq b.$$

Show that

$$r(t) \leq \int_a^t k(s)f(s)e^{\int_s^t k(u)du}ds + f(t), \ a \leq t \leq b.$$

2.(15%) Let E be a normed vector space, $W \subset \mathbf{R} \times E$ an open set, and $f, g : W \to E$ continuous. Suppose that for all $(t, x) \in W$, $|f(t, x) - g(t, x)| < \varepsilon$. Let K be a Lipschitz constant in x for $f(t, x)$. If $x(t), y(t)$ are solutions to $x' = f(t, x)$ and $y' = g(t, y)$, respectively, on some interval J, and $x(t_0) = y(t_0)$. Show

$$|x(t) - y(t)| \leq \frac{\varepsilon}{K}\{e^{K|t-t_0|} - 1\}.$$

3.(15%) Let I be an finite interval, $g_i \in C(I), k_i \in C^1(I)$ for $i = 1, 2$ with $g_1 < g_2$ and $k_1 > k_2 > 0$ on $\bar{I}$. Let ϕ_i be a solution on I of $(k_i y')' + g_i y = 0$ and let t_1 and t_2 be two consecutive zeros of ϕ_1. Show that ϕ_2 has at least one zero in the interval (t_1, t_2).

4. (10%) If a nontrivial solution ϕ of $y'' + (A + B\cos 2t)y = 0$ has $2n$ zeros in $(-\pi/2, \pi/2)$ and if $A, B > 0$, show that $A + B \geq (2n-1)^2$.

5.(20%) Write the planar system

$$\frac{dx}{dt} = -y - x(\mu - (x^2 + y^2 - 1)^2),$$
$$\frac{dy}{dt} = x - y(\mu - (x^2 + y^2 - 1)^2),$$

in polar coordinates and show that for all $\mu > 0$ there are two one-parameter families of periodic orbits given by $\gamma_\mu^{\pm}(t) = \sqrt{1 \pm \mu^{1/2}}(\cos t, \sin t)^T$ with parameter μ. Discuss the phase portraits and draw the bifurcation diagram.