Feb. 2009

Ordinary Differential Equations

1. (20%) Consider the following initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \;\; f: D \subset R \times R^n \to R^n \\ x(t_0) = x_0. \end{cases} \tag{1}$$

Prove the following statements respectively :

(a) Let $f \in C(D)$ and $(t_0, x_0) \in D$. Then (1) has a solution on an interval $I = [t_0 - c, t_0 + c]$ for some $c > 0$.

(b) If, in addition, $f(t,x)$ satisfies a Lipschitz condition in x on D, then for any (t_0, x_0) in D, there exists a unique solution $x(t, t_0, x_0)$ of (1) passing through (t_0, x_0).

(c) Let $\phi(t, t_0, \xi_0)$ be the unique solution of

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = \xi_0. \end{cases} \tag{2}$$

on $J = [a, b]$. Assume f_x exists and is continuous at all points $(t, \phi(t, t_0, \xi_0)), t \in J$. Then the system $E_{\tau,\xi} : \frac{dx}{dt} = f(t,x), x(\tau) = \xi$ has a unique solution $\phi(t, \tau, \xi)$ for (τ, ξ) near (t_0, ξ_0). Moreover $\phi_\xi(t, t_0, \xi_0)$ exists $(t \in J)$ and satisfies the linearized homogeneous equation

$$\begin{cases} \dfrac{dY}{dt} = f_x(t, \phi(t, t_0, \xi_0))Y \\ Y(t_0) = I. \end{cases}$$

2. (20%) Consider the Lotka-Volterra two species competition model

$$\begin{cases} \dfrac{dx}{dt} = \gamma_1 x(1 - \dfrac{x}{K_1}) - \alpha_1 xy \\ \dfrac{dy}{dt} = \gamma_2 y(1 - \dfrac{y}{K_2}) - \alpha_2 xy \\ \gamma_1, \gamma_2, K_1, K_2, \alpha_1, \alpha_2 > 0 \\ x(0) > 0, y(0) > 0. \end{cases} \tag{3}$$

For various possible cases, do the following :

(a) Do the stability analysis for each equilibrium with nonnegative components.

(b) Find the stable manifold of each saddle point.

3. (20%) Consider the van der Pol equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0. \tag{4}$$

Prove the following statements.

(a) Equation (4) has a unique asymptotically stable limit cycle Γ for every $\epsilon > 0$. (State the theorem used.)

(b) Let $D = \{(x, y) | x^2 + y^2 < 3\}$. Then $\Gamma \not\subset D$. (State the theorem used.)