4. (20%) Consider the equation of simple pendulum with damping

$$\theta'' + \beta\theta' + k^2 \sin\theta = 0, \quad \beta > 0, k > 0, \tag{5}$$

and its equivalent system

$$\begin{cases} \theta' = \psi \\ \psi' = -k^2 \sin\theta - \beta\psi. \end{cases} \tag{6}$$

(a) Prove $\lim\limits_{t\to\infty} \theta'(t) = 0$ by using the decreasing property of the energy function for system (6).

(b) Sketch the approximate orbits of system (6) in the phase plane.

5(a). (5%) State the Poincare-Bendixson Theorem.

(b) (15%) Use the Poincare-Bendixson to prove the following system possesses a limit cycle:

$$\begin{cases} \dfrac{dx}{dt} = \gamma x(1 - \dfrac{x}{K}) - \dfrac{mx}{a+x}y \\ \dfrac{dy}{dt} = (\dfrac{mx}{a+x} - d)y, \qquad \gamma, K, m, a, d > 0. \\ x(0) > 0, y(0) > 0, \end{cases} \tag{7}$$