歸檔

NCU Qualify Exam for Ordinary Differential Equations

09/01/2009

1. (a) (10 pts) Let $f(t,x)$ be continuous on

$$S = \{(t,x) : |t - t_0| \leq a,\ |x - x_0| \leq b\}$$

and let $f(t,x)$ satisfy a Lipschitz condition in x with Lipschitz constant L. Let $M \equiv \max\{|f(t,x)| : (t,x) \in S\}$. Show by contraction principle that there exists a unique solution of initial value problem

$$\frac{dx}{dt} = f(t,x),$$
$$x(t_0) = x_0$$

on $I = \{t : |t - t_0| \leq \alpha\}$, where $\alpha < \min\{a, \frac{b}{M}, \frac{1}{L}\}$.

(b) (10 pts) Consider the equation

$$\dot{x} = \phi(t)\frac{x}{1+t^2+x^2} = f(t,x)$$

for all $t, x,\ \phi(t) > 0$, and $\int^{\infty} \phi(t)dt < \infty$. Show that every solution approaches a constant as $t \to \infty$.

2. (a) (10 pts) We consider Hill's equation

$$x'' + (\alpha + p(t))x = 0,$$

where $\alpha > 0$ and $p(t)$ is periodic with period T. Rewrite Hill's equation as a linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A(t)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\Phi(t)$ be the fundamental matrix of the linear system above. Also let

$$\Phi(0) = I,\ -2 < \mathbf{tr}(\Phi(t)) < 2.$$

Show that the solutions of Hill's equation are bounded.